Conformal maps of some Fortran 2008 complex intrinsics, specifically on branch cuts

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1 Introduction

All real and complex variables and constants are definied with kind `real64` from the intrinsic module `iso_fortran_env`, corresponding to IEEE binary64 type. $z = x + iy$ is a complex variable mapped onto w. Gnuplot is used for plotting.

2 LOG

LOG(X) for zero x is not allowed in Fortran 2008. We check values of LOG for x in -HUGE, -TINY and -1. The top and the bottom boundaries of the cut must be mapped to $y = +\pi$ and $y = -\pi$ respectively, see Fig. 1.

Fig. 2 shows that log maps a ring into a rectangle. The top boundary of the cut, AJ, must be mapped onto the top side of the rectangle. The bottoom boundary of the cut, EF, must be mapped onto the bottom side of the rectangle.

3 SQRT

SQRT can take any complex value, including zero. So checking the values for x from -HUGE to 0. The top and the bottom boundaries of the cut must be mapped to $\theta = +\pi/2$ and $\theta = -\pi/2$ respectively, see Fig. 3. The map of $w = \sqrt{z}$ is shown in Fig. 4. Note that $y = +0$ is mapped onto $\theta = +\pi/2$,

4 ASIN

The map of ASIN is easy to understand. Each cut is opened wider and wider until it becomes a straight line. So the ASIN map is band between $x = -\pi/2$ to $x = +\pi/2$. Along the imaginary axis the band stretches to plus and minus infinity, See Fig. 5. Fig. 6 shows that points on the branch cuts with $y = +0$ are mapped onto points with $y \geq 0$.

5 ACOS

ACOS is defined everywhere. The branch cuts are on real axis, from $-\infty$ to -1 and from $+1$ to $+\infty$, see Fig. 7.

For $x \leq -1$, the top boundary of the cut, $y = +0$, is mapped onto $(\pi, b \leq 0)$, and the bottom boundary of the cut, $y = -0$, is mapped onto $(\pi, b \geq 0)$.

For $x \geq +1$, the top boundary of the cut, $y = +0$, is mapped onto $(0, b \leq 0)$, and the bottom boundary of the cut, $y = -0$, is mapped onto $(0, b \geq 0)$.

Fig. 8 shows a map of a square. Note that the $y = 0$ line is mapped onto $(x, y \leq 0)$.

To visualise the ACOS map first visualise the ASIN map. The ACOS map can be obtained from the ASIN map in 3 steps: (1) flip the ASIN map about the real axis, then (2) flip the ASIN map about the imaginary axis, then (3) shift the ASIN map along x so that the map of the left branch cut starts at $x = 0$.

In Fig. 8 the region of z which is mapped onto w is not symmetric with respect to $x = 0$ axis. This lack of symmetry is used to highlight that points with $\mathrm{Re}z \geq 0$ are mapped onto points with $\mathrm{Re}w \leq 0$.

6 ATAN

Atan has 2 branch cuts along the imaginary axis, from $+i$ to $+\infty$, and from $-i$ to $-\infty$. Fig. 9 shows `atan` on branch cuts. Fig. 10 shows a map of `atan`.

7 ASINH

Figs. 11 and 12 show that the ASINH map is a band between $y = -\pi/2$ and $y = \pi/2$. Along the real axis the band stretches from minus to plus infinity. Points on the branch cut with $x = +0$ are mapped onto points with $x \geq 0$.

8 ACOSH

A single branch cut along the real axis from $x = -1$ to $-\infty$. Checking at points -HUGE, -1, 0, 1 on both sides of the cut, Fig. 13.

Fig. 14 shows that without -0 only the top boundary of the branch cut is mapped. The symmetry is lost.

9 ATANH

Atanh has 2 branch cuts along the real axis, from $+1$ to $+\infty$, and from -1 to $-\infty$. Fig. 15 shows `atanh` on branch cuts. Fig. 16 shows a map of `atanh`.

10 $w = \frac{1}{2}(z + \mathbf{copysign}(1, \mathbf{RE}(z))\sqrt{z^2 - 4})$

This function is the inverse of

$$z = w + \frac{1}{w} \tag{1}$$

which is used often in fracture mechanics to map a unit circle (or an infinite plane with a unit circle cut out) on to a an infinite plane with a centre crack.

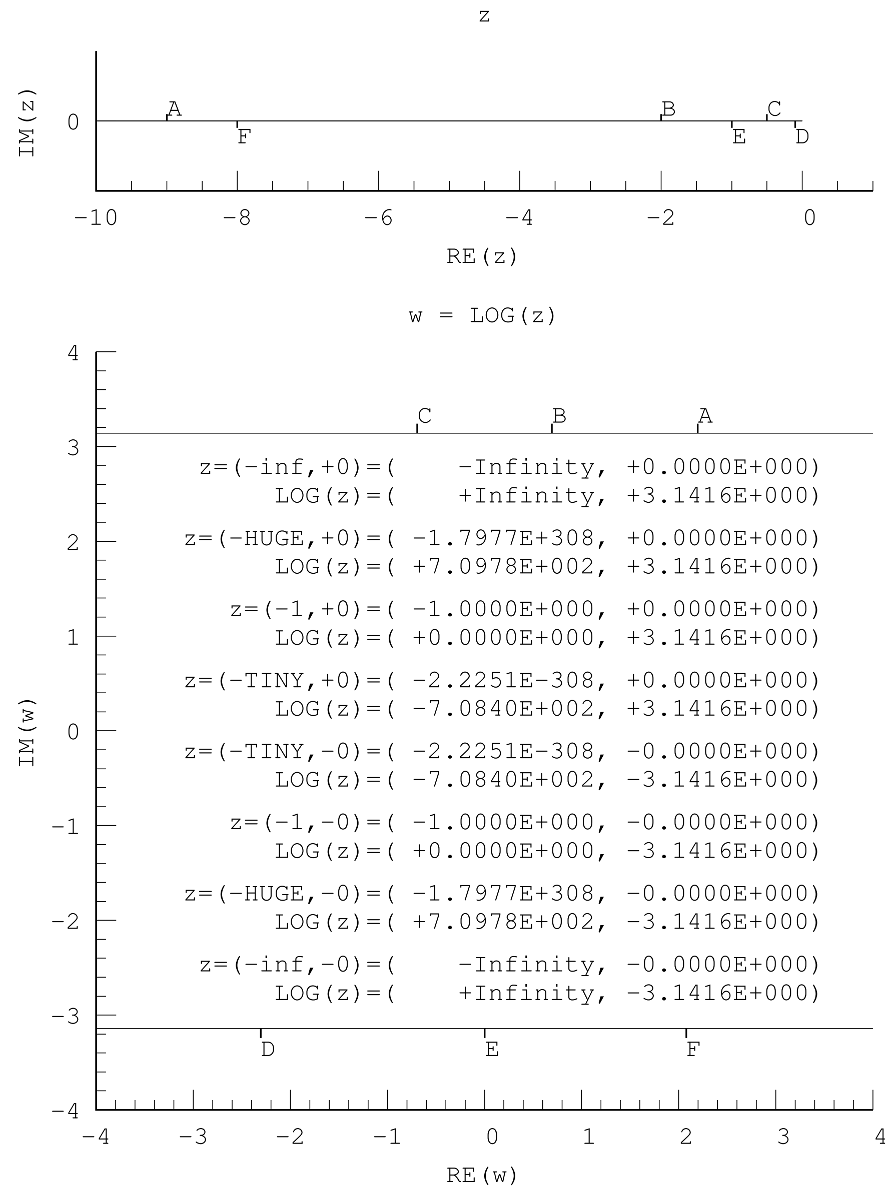


Figure 1: Map of the branch cut of $w = \log z$. Points A, B, C are on the top boundary of the cut, $y = +0$. Points D, E, F are on the bottom boundary of the cut, $y = -0$.

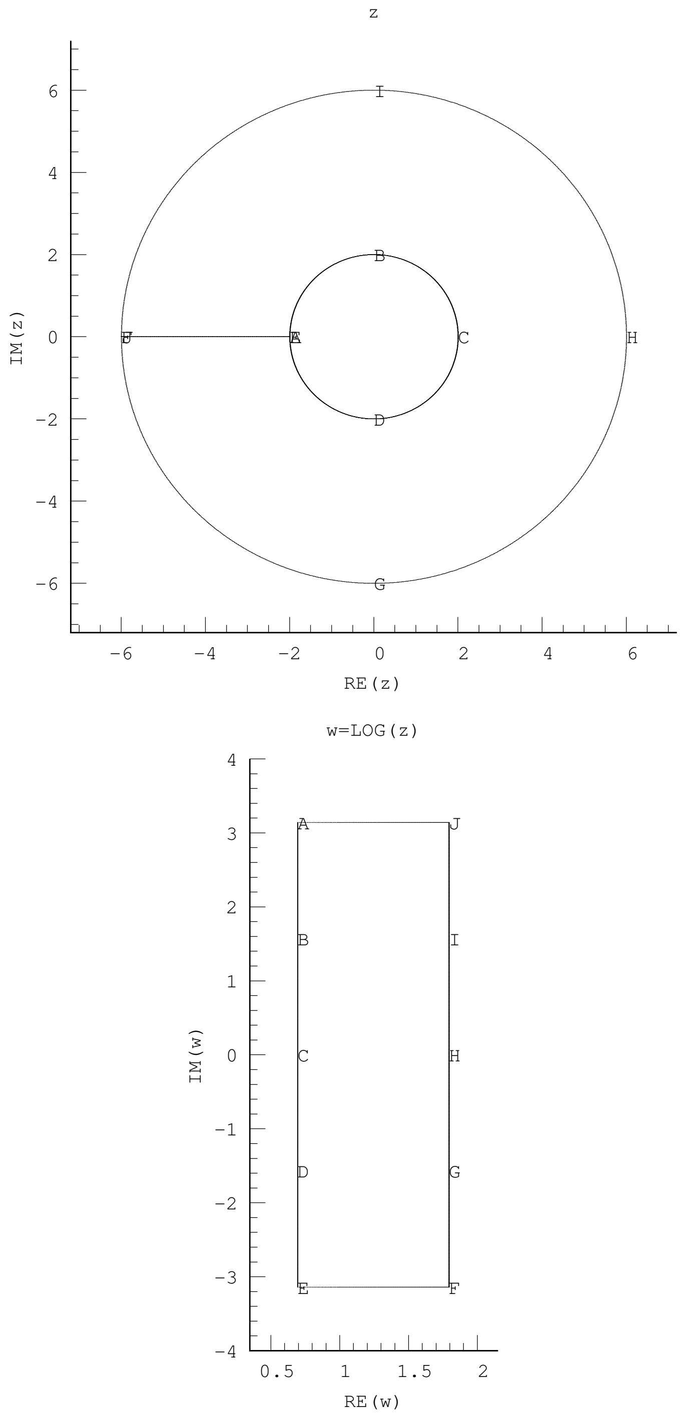


Figure 2: Map of $w = \log z$. Points A, J are at the top of the cut, $y = +0$. Points E, F are at the bottom of the cut, $y = -0$.

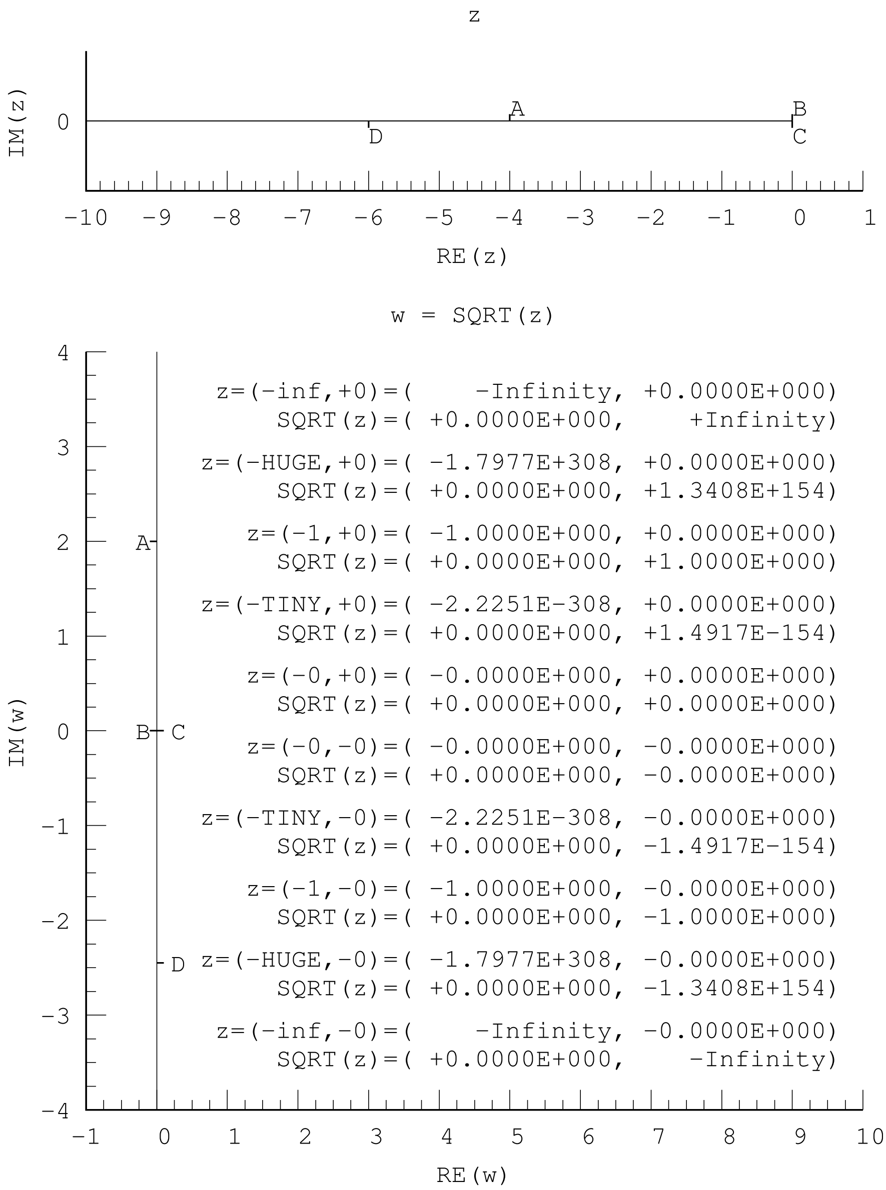


Figure 3: Map of the branch cut of $w = \sqrt{z}$. Points A, B are on the top boundary of the cut, $y = +0$. Points C, D are on the bottom boundary of the cut, $y = -0$. Points B and C are at $x = 0$.

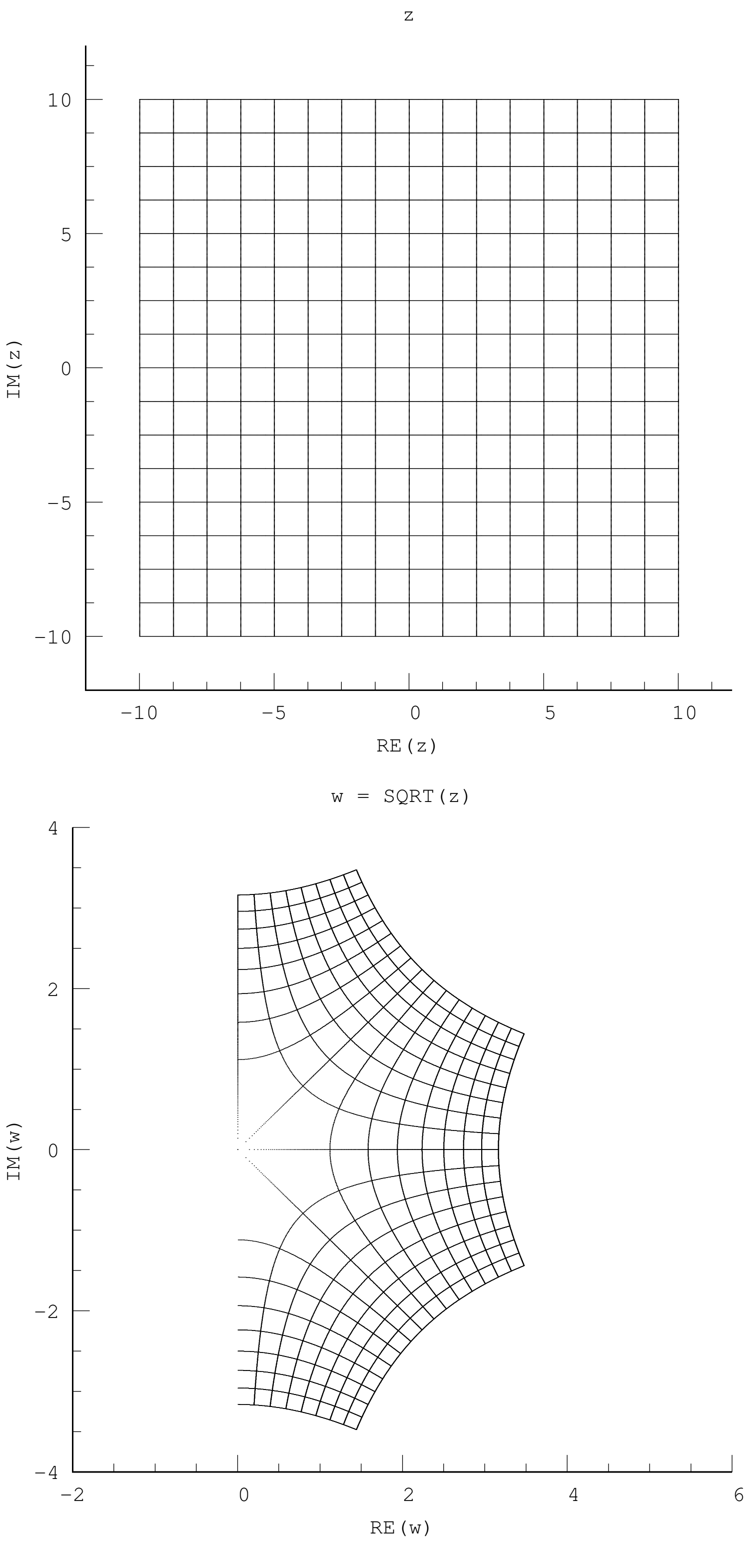


Figure 4: Map of $w = \sqrt{z}$.

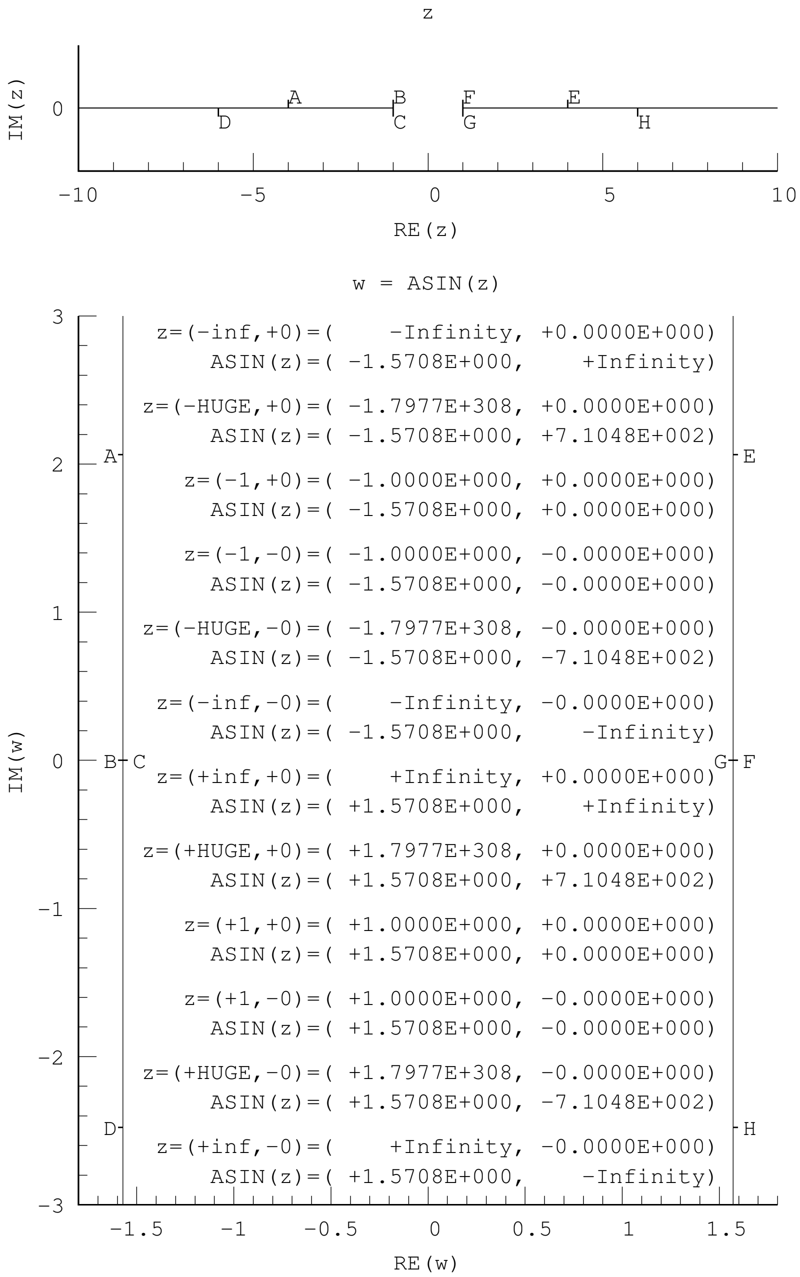


Figure 5: Map of $w = \texttt{asin} z$. Points A, B, E, F are on the top boundary of the cut, $y = +0$. Points C, D, G, H are on the bottom boundary of the cut, $y = -0$. Points B and C are at $x = -1$. Points F and G are at $x = +1$.

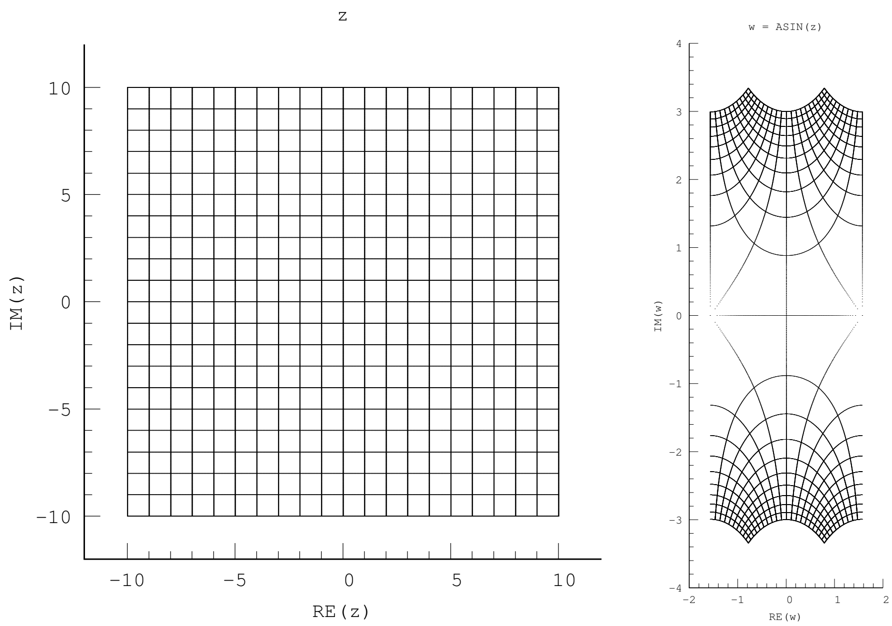


Figure 6: Map of $w = \mathtt{asin} z$.

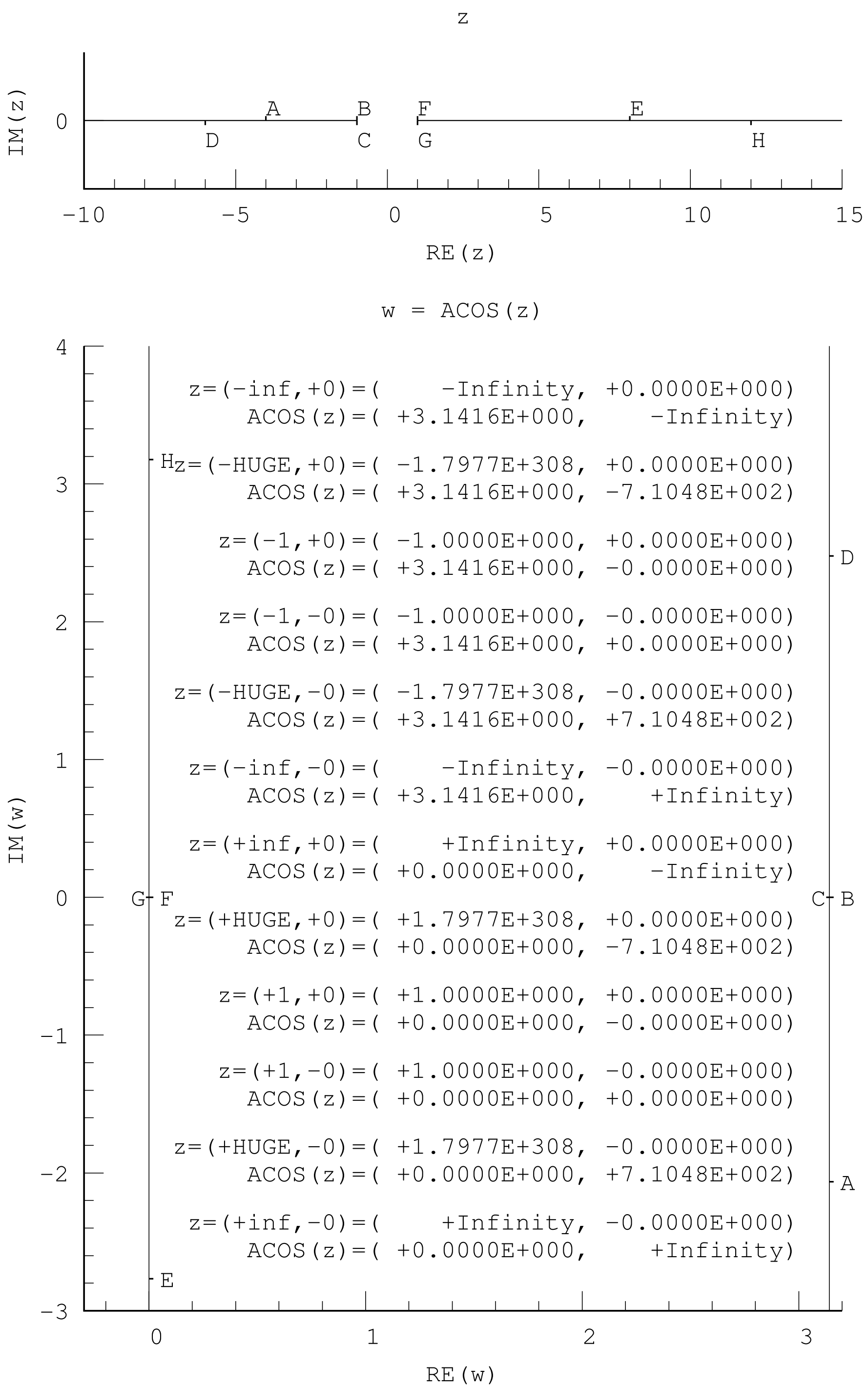


Figure 7: Map of $w = \texttt{acos}z$. Points A, B, E, F are on the top boundary of the cut, $y = +0$. Points C, D, G, H are on the bottom boundary of the cut, $y = -0$. Points B and C are at $x = -1$. Points F and G are at $x = +1$.

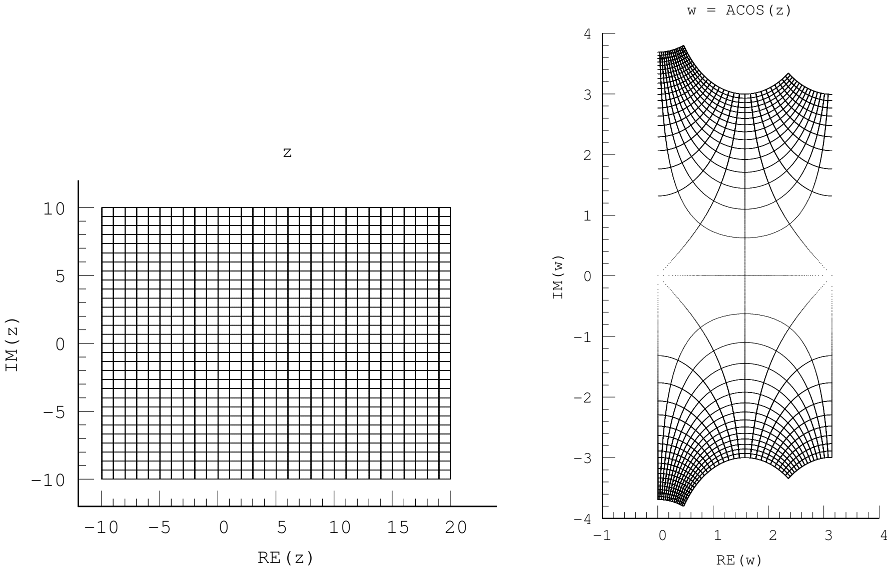


Figure 8: Map of $w = \texttt{acos}z$.

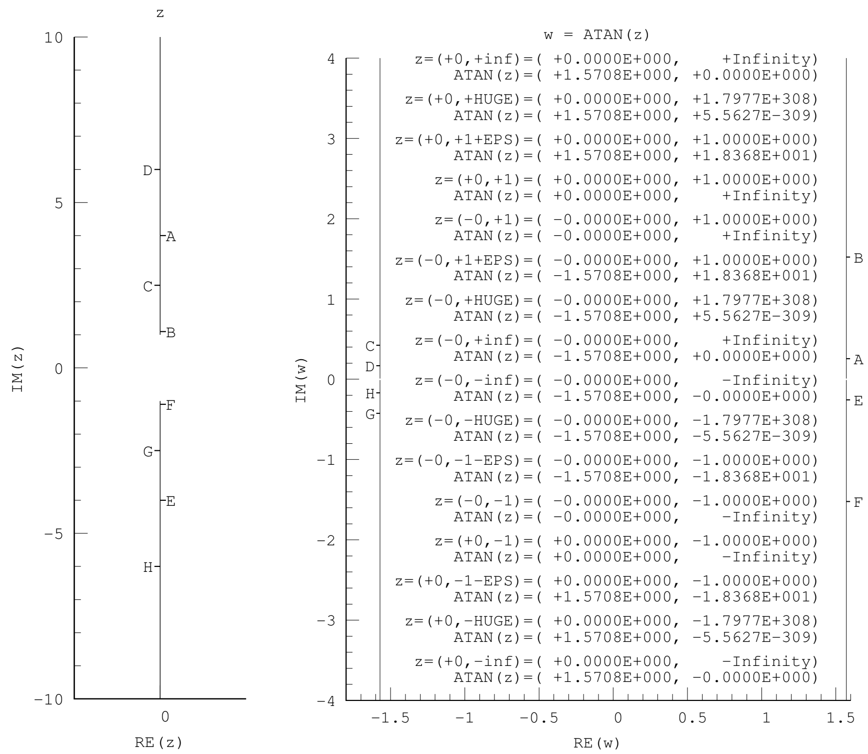


Figure 9: Map of $w = \texttt{atan}z$. Points A, B, E, F are at $x = +0$. Points C, D, G, H are at $x = -0$.

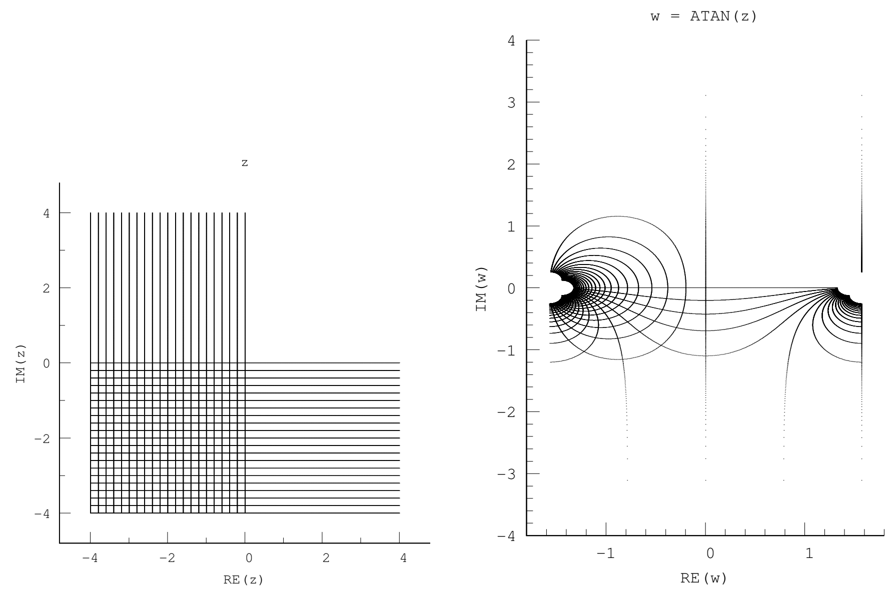


Figure 10: Map of $w = \texttt{atan} z$.

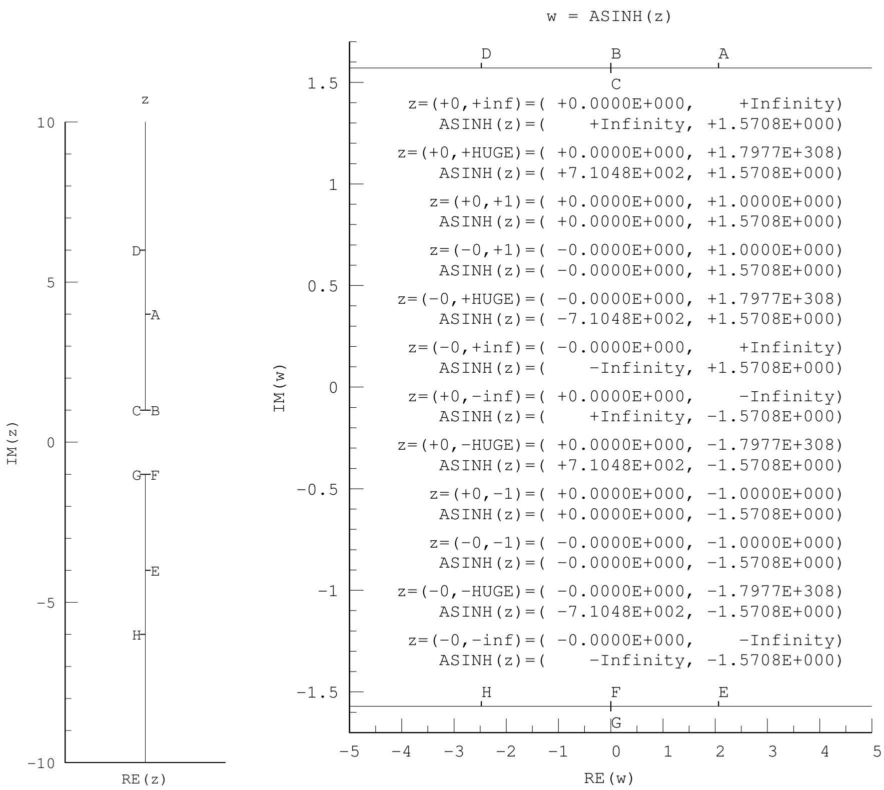


Figure 11: Map of $w = \texttt{asinh} z$. Points A, B, E, F are at $x = +0$. Points C, D, G, H are at $x = -0$. Points B and C are at $y = +1$. Points F and G are at $y = -1$.

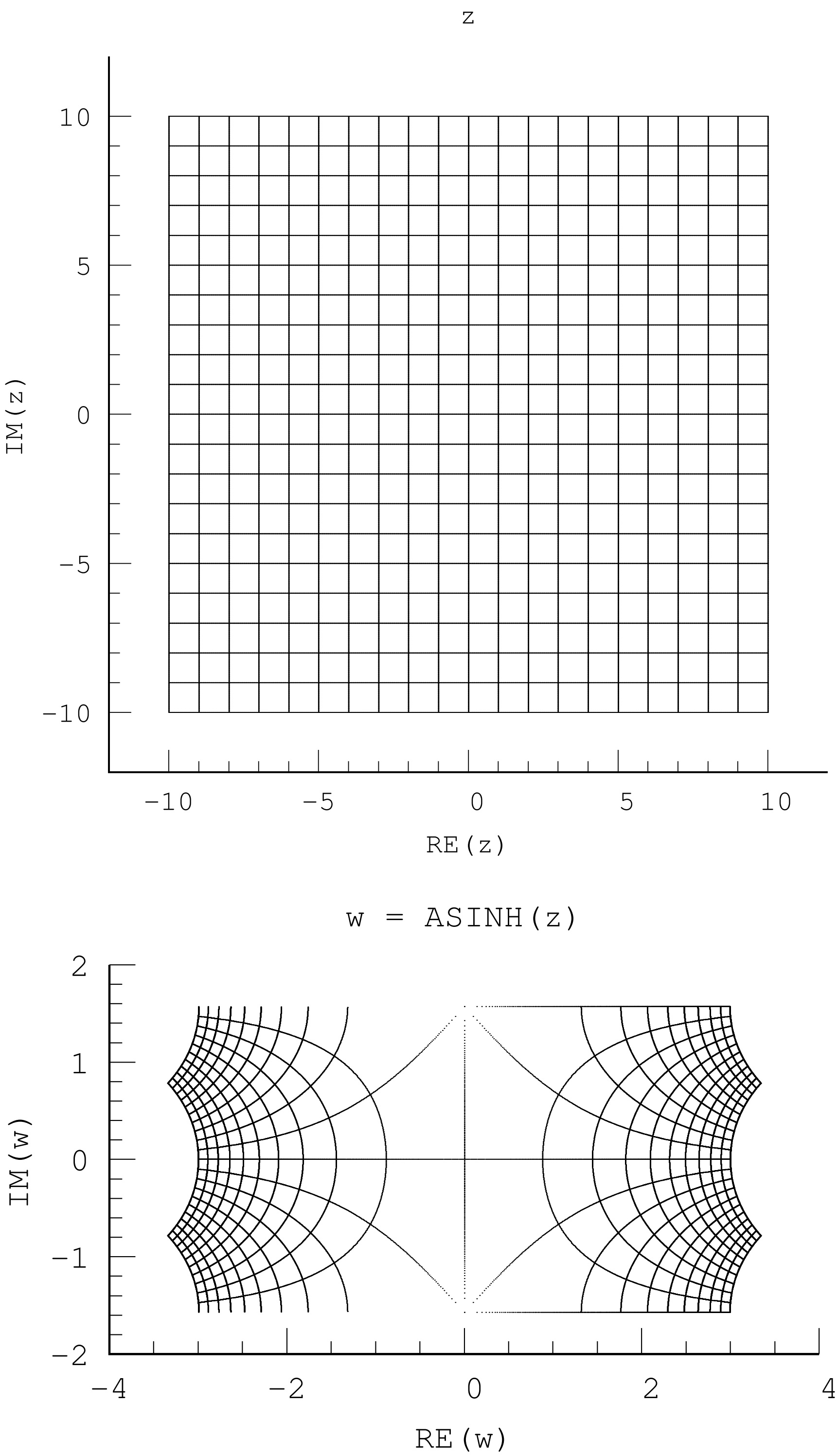


Figure 12: Map of $w = \mathtt{asinh} z$.

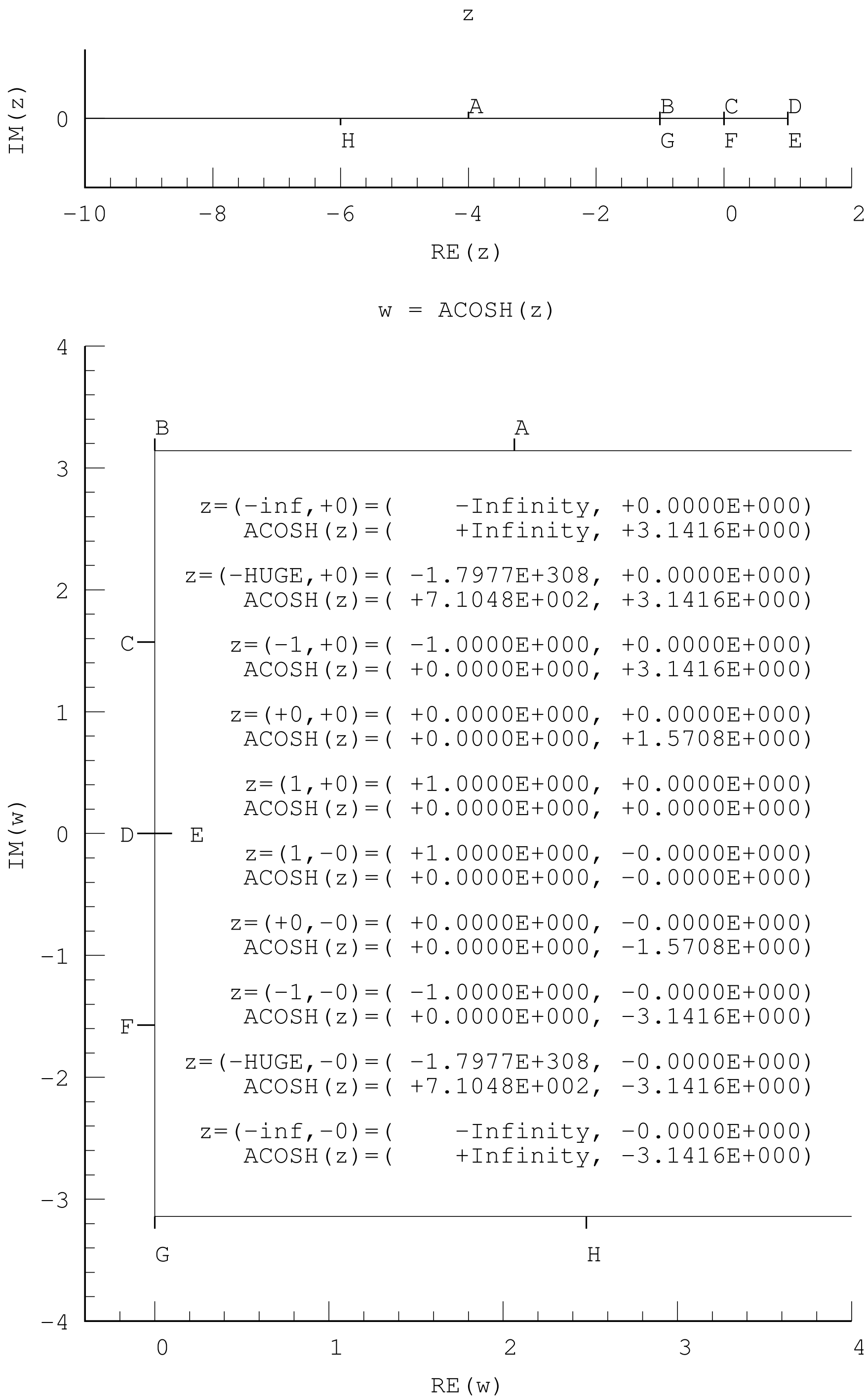


Figure 13: Map of $w = \texttt{acosh} z$. Points A, B, C, D are at $y = +0$. Points E, F, G, H are at $y = -0$.

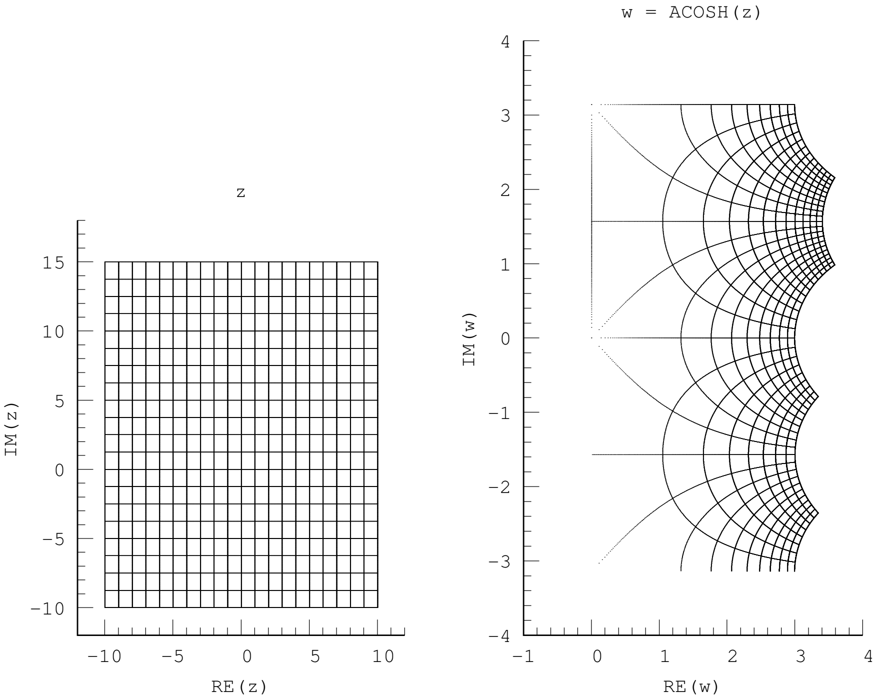
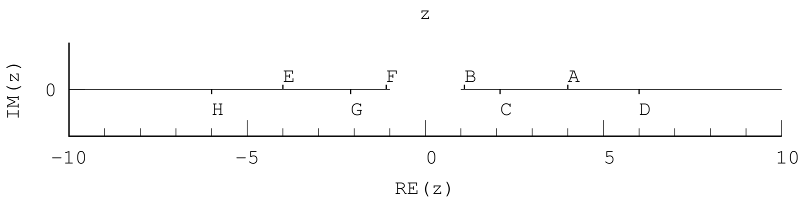


Figure 14: Map of $w = \texttt{acosh} z$.



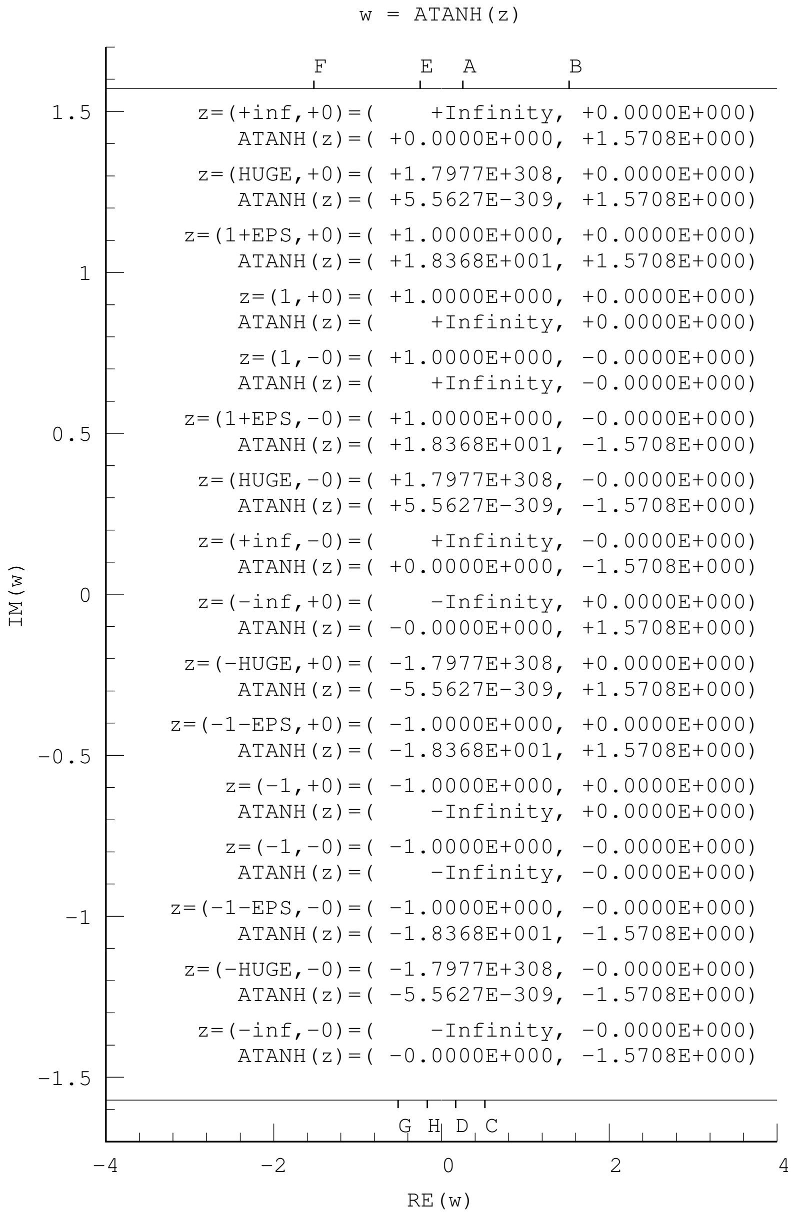


Figure 15: Map of $w = \mathtt{atanh} z$. Points A, B, E, F are at $y = +0$. Points C, D, G, H are at $y = -0$.

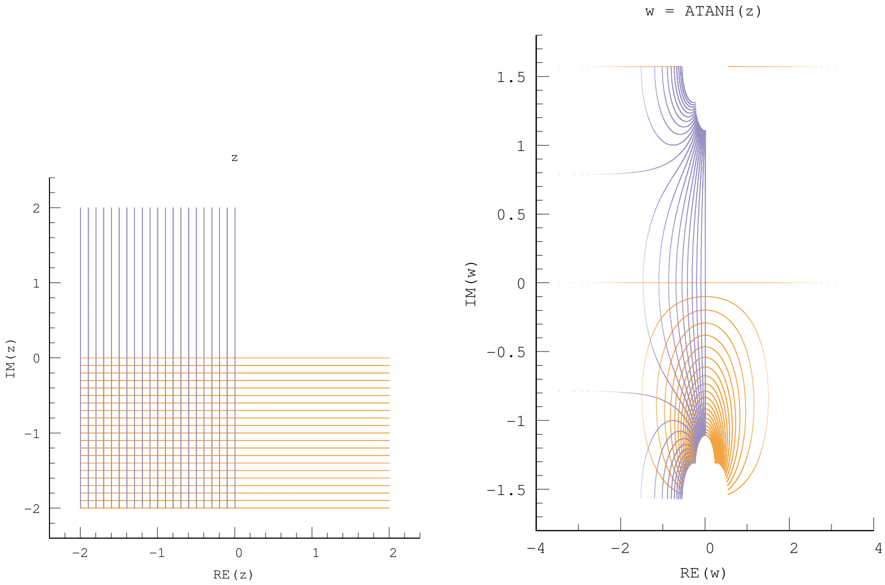


Figure 16: Map of $w = \texttt{atanh} z$.

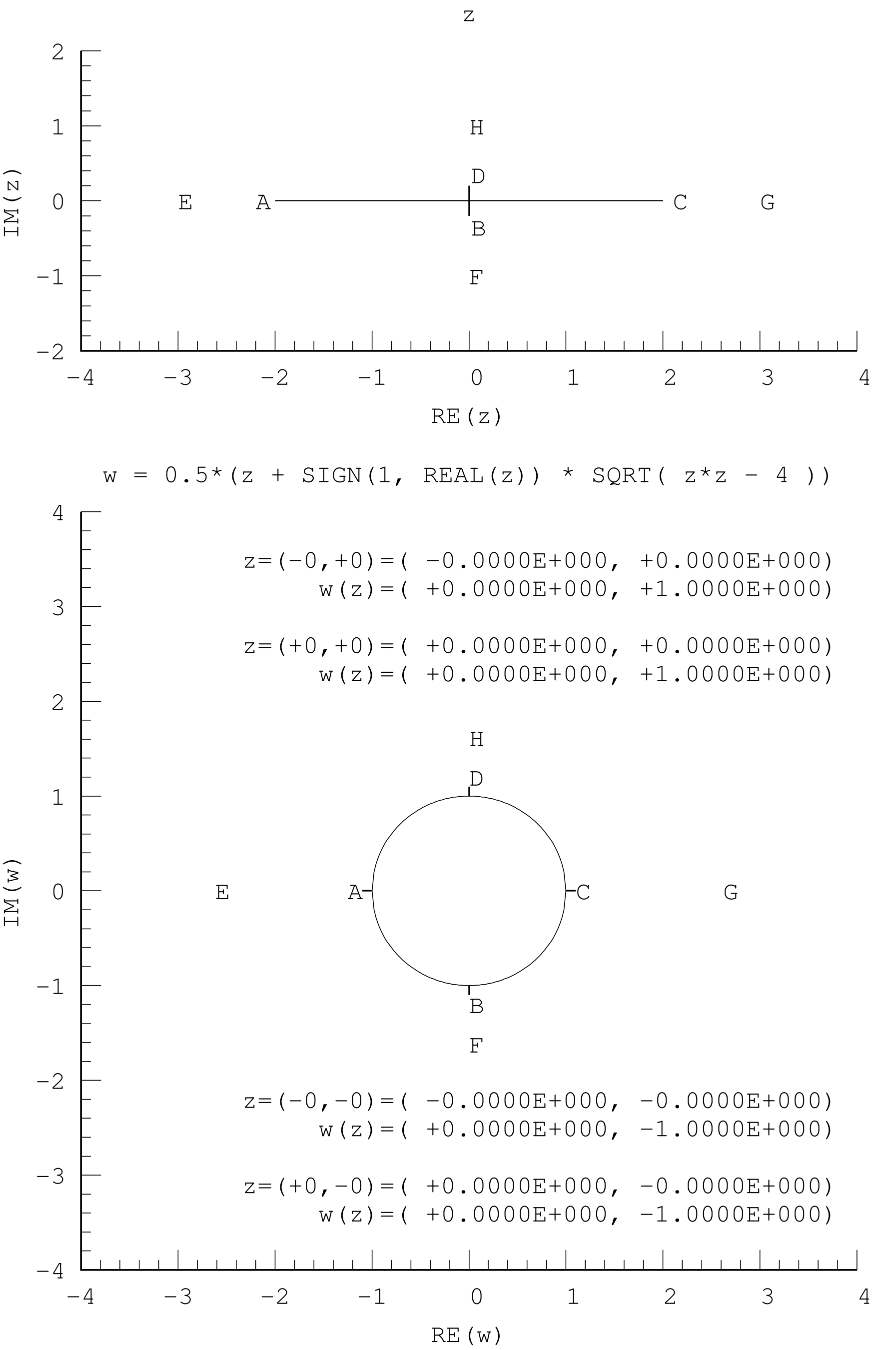


Figure 17: Map of $w = \frac{1}{2}(z + \text{copysign}(1, \text{RE}(z))\sqrt{z^2 - 4})$.

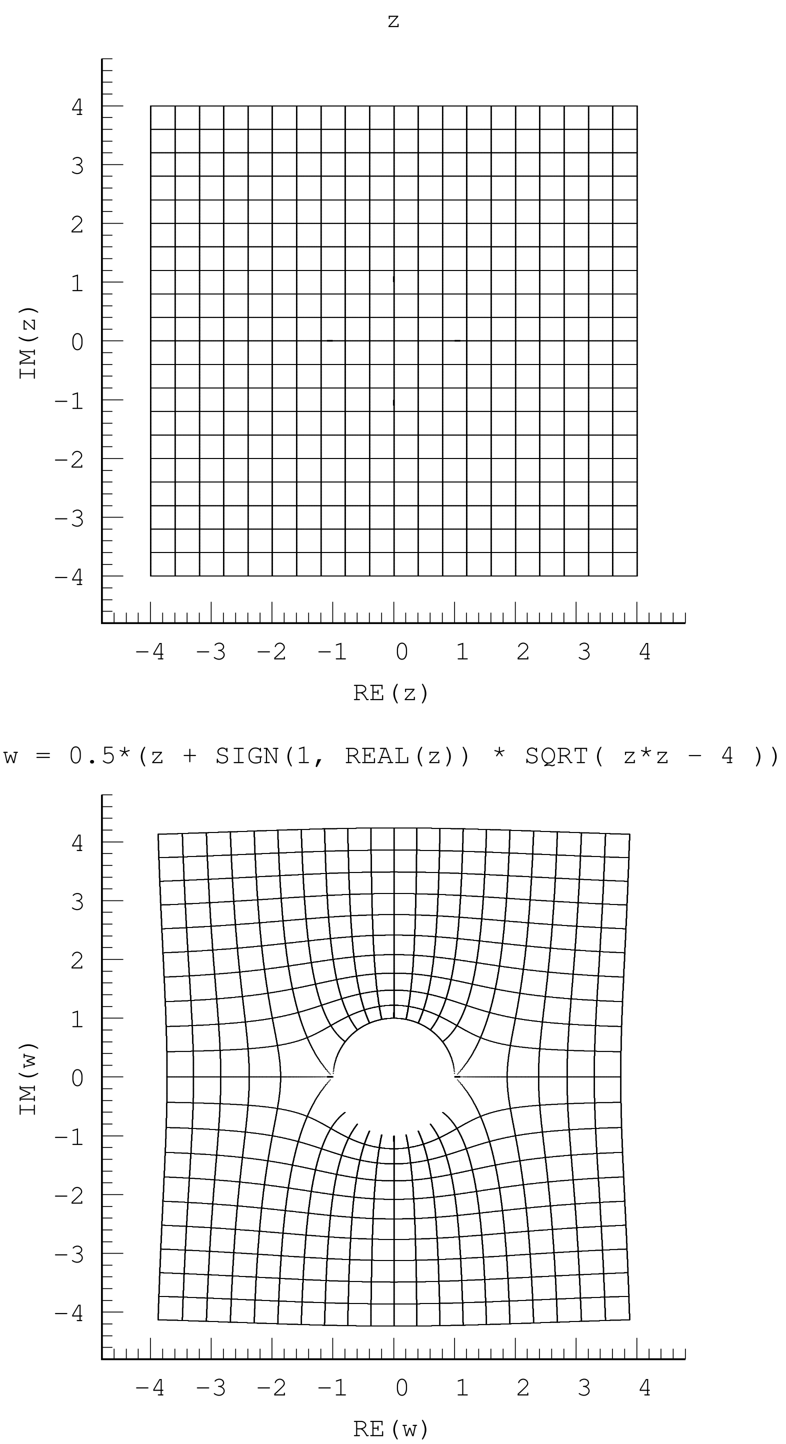


Figure 18: Map of $w = \frac{1}{2}(z + \mathrm{copysign}(1, \mathrm{RE}(z))\sqrt{z^2 - 4})$.