

# Fortran 2008: conformal mapping, branch cuts and minus zero

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## 1. Introduction

All real and complex variables and constants are defined with kind `real64` from the intrinsic module `iso_fortran_env`.

In the following  $z = x + iy$  is a complex variable mapped onto  $w$  with one or the other complex function.

Gnuplot is used for plotting.

## 2. Intrinsic functions

There are several intrinsic functions in Fortran 2008 which take complex arguments and distinguish in some way positive and negative zero.

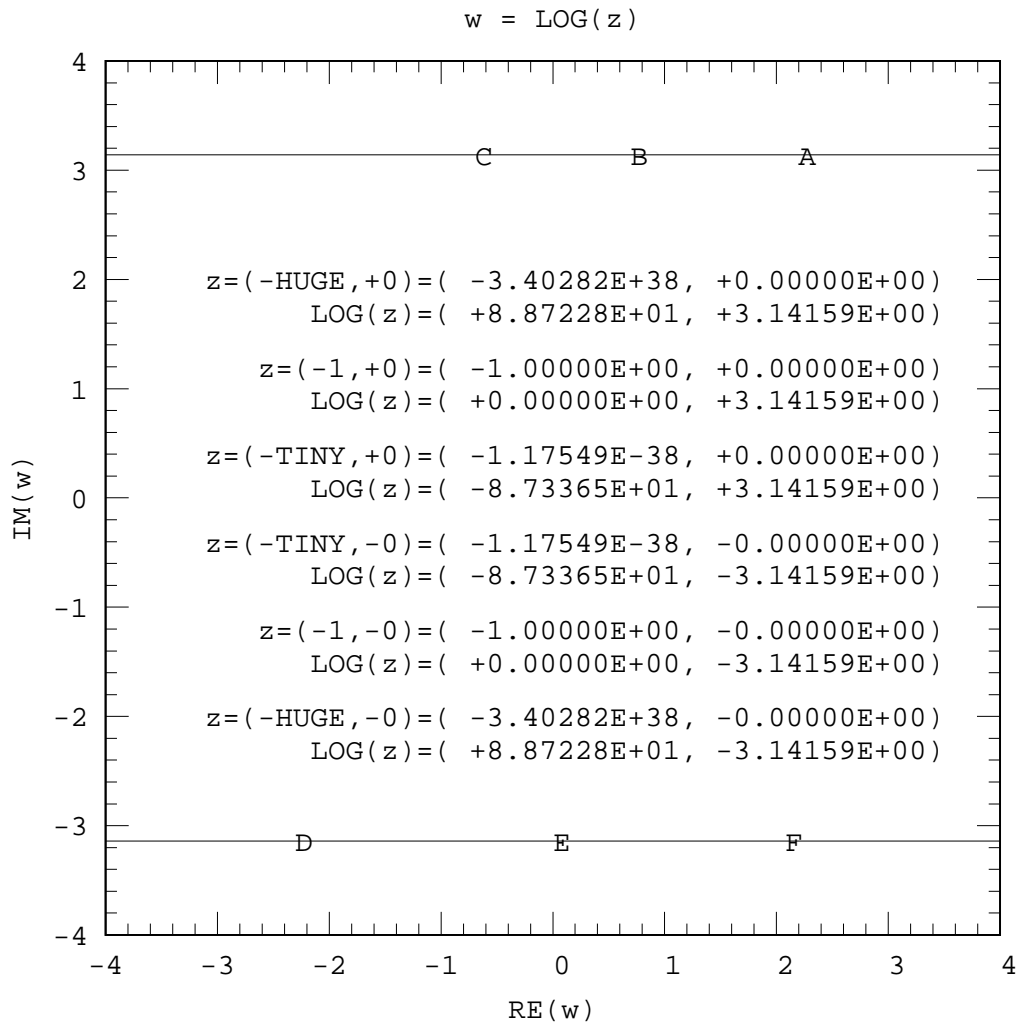
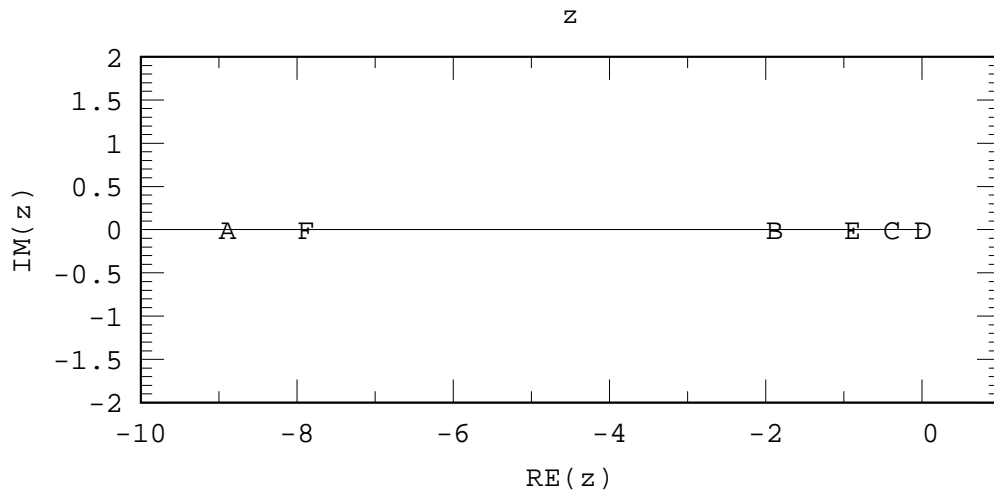
### 2.1. LOG

`LOG(z)` for  $z = 0 + i0$  is not allowed in Fortran 2008. We check values of `LOG` for  $x$  in `-HUGE`, `-TINY` and `-1`.

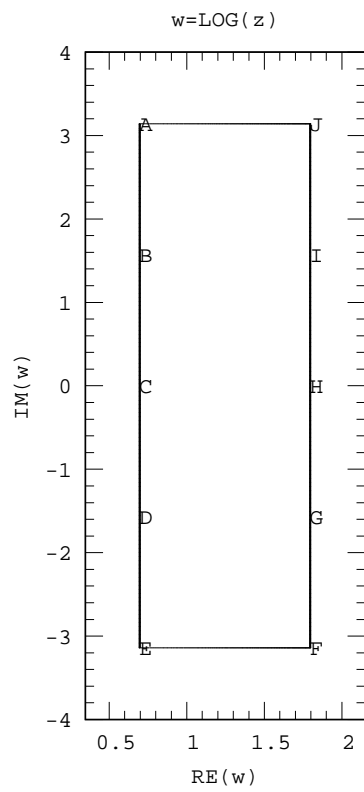
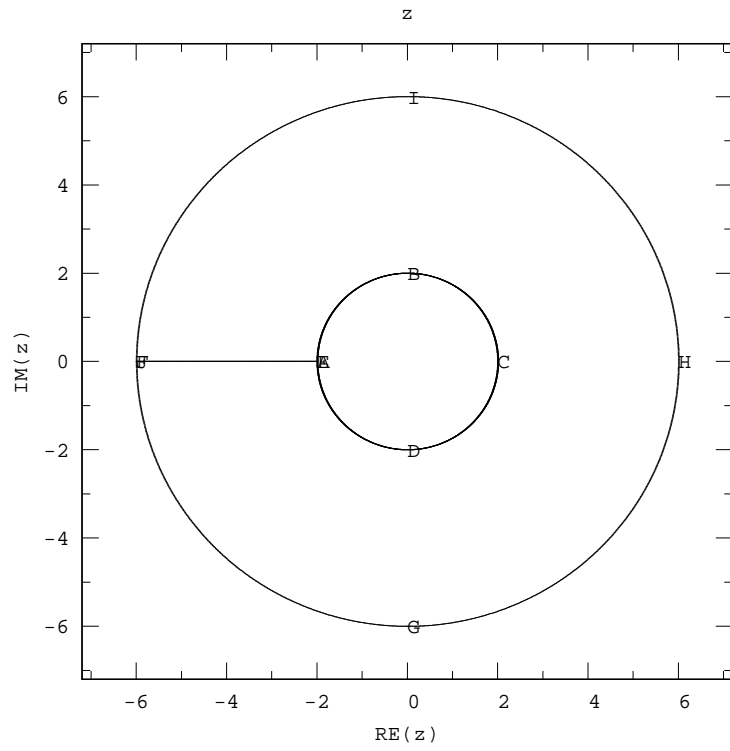
Points A, B, C are at  $y = +0$ . These must be mapped to  $y = +\pi$ . Points D, E, F are at  $y = -0$ . These must be mapped to  $y = -\pi$ .

`LOG` maps a ring into a rectangle. The top boundary of the cut, AJ, must be mapped onto the top side of the rectangle. The bottom boundary of the cut, EF, must be mapped onto the bottom side of the rectangle.

2.1.1. LOG branch cut map



### 2.1.2. LOG map



## 2.2. SQRT

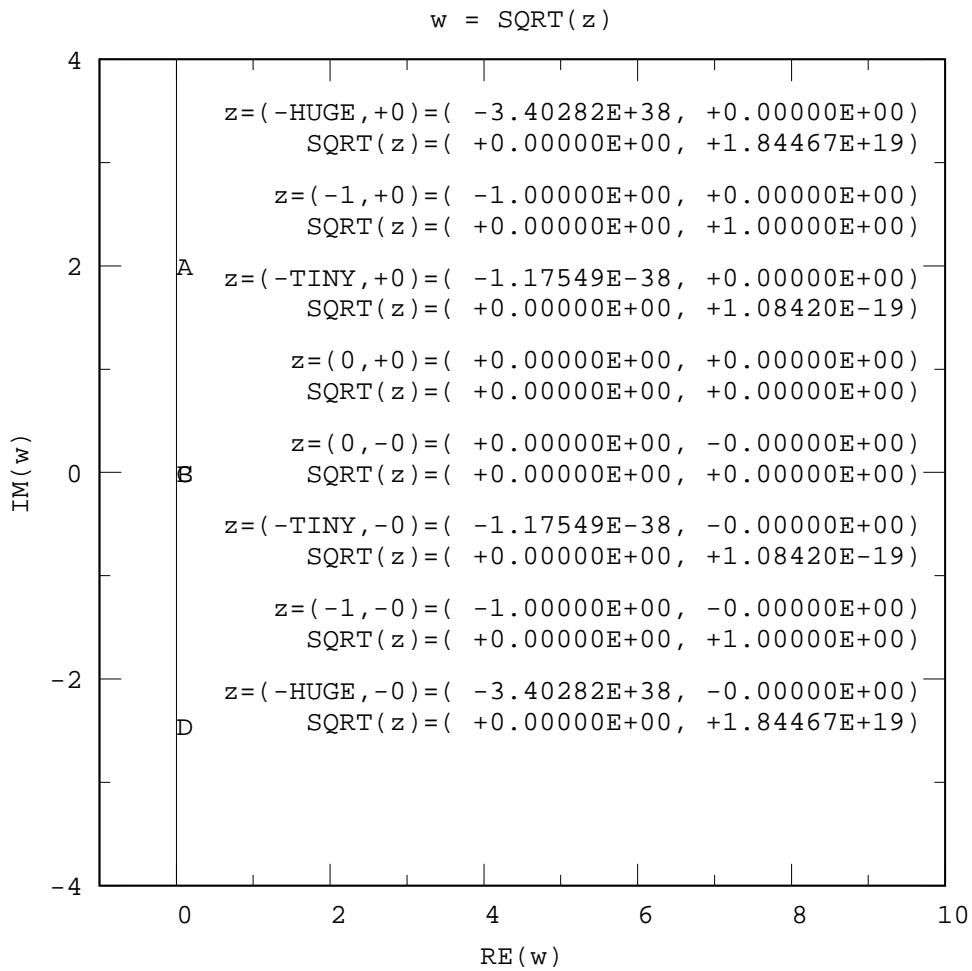
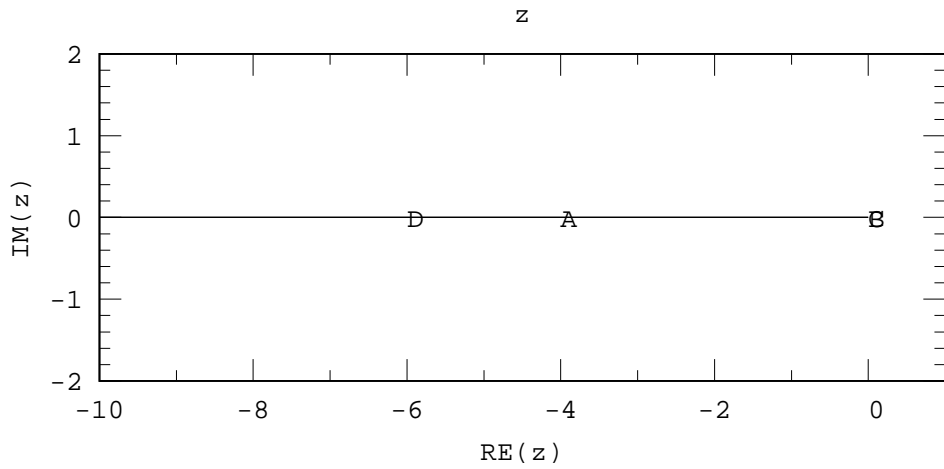
SQRT can take any complex value, including zero. So checking the values for  $x$  from  $-\text{HUGE}$  to  $0$ .

A square root identity is  $\sqrt{z} = \exp(\frac{1}{2} \log z)$ . However, since  $\log z$  for  $z = 0 + i0$  is not defined, this seems to fail at  $z = 0 + i0$ . Yet,  $\sqrt{0 + i0} = 0$ . A way forward seems to define a value to  $\log(0 + i0)$ . Any phase can be used within its principal value, i.e.  $-\pi \leq \text{Arg}z \leq +\pi$ . Zero seems an obvious easy choice. The magnitude of  $\log(0 + i0)$  must be  $-\infty$ . So one possible definition is:  $\text{LOG}(z)$  for  $z = 0 + i0$  is  $(-\infty, 0)$ . Then  $\text{EXP}(\text{LOG}(0)) = 0$ .

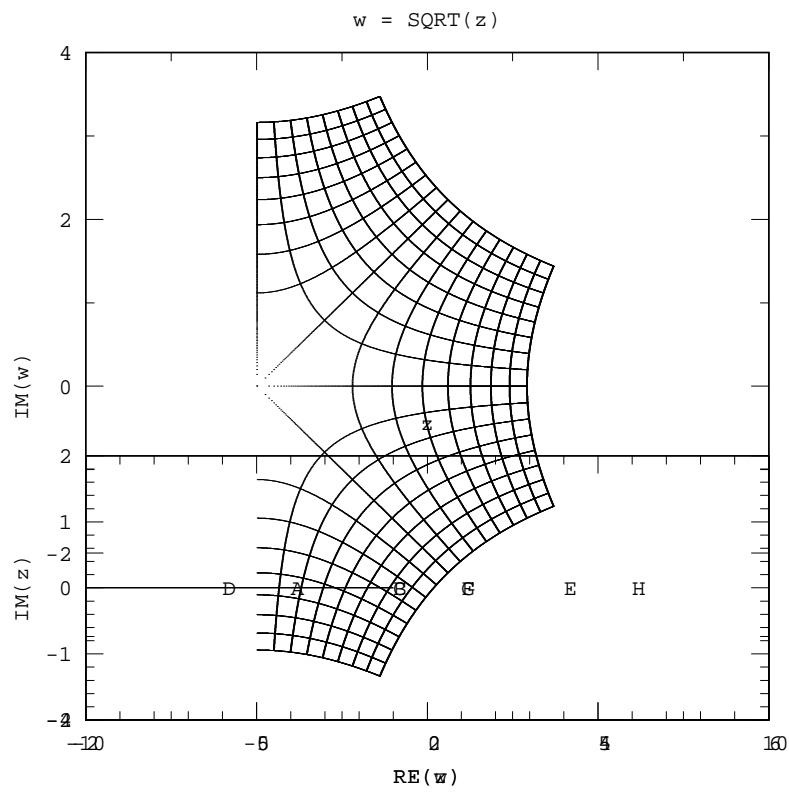
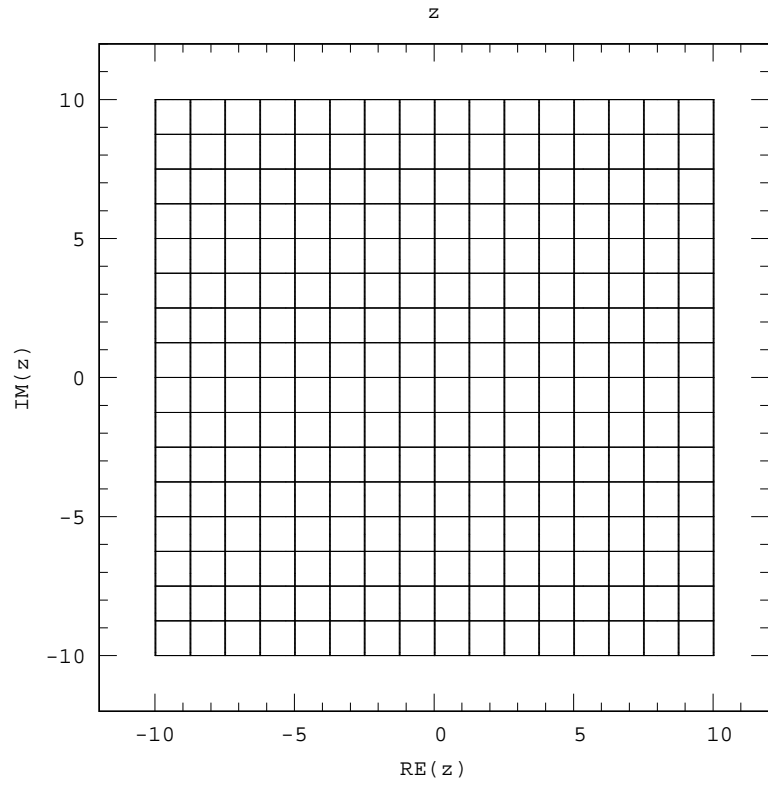
Points A, B are at  $y = +0$ . These must be mapped to  $y \geq +0$ . Points C, D are at  $y = -0$ . These must be mapped to  $y \leq -0$ . B is mapped from  $0 + i0$  to  $0 + i0$ . C is mapped from  $0 - i0$  to  $0 - i0$ .

The top and the bottom boundaries of the cut must be mapped to  $\theta = +\pi/2$  and  $\theta = -\pi/2$  respectively.

### 2.2.1. SQRT branch cut map



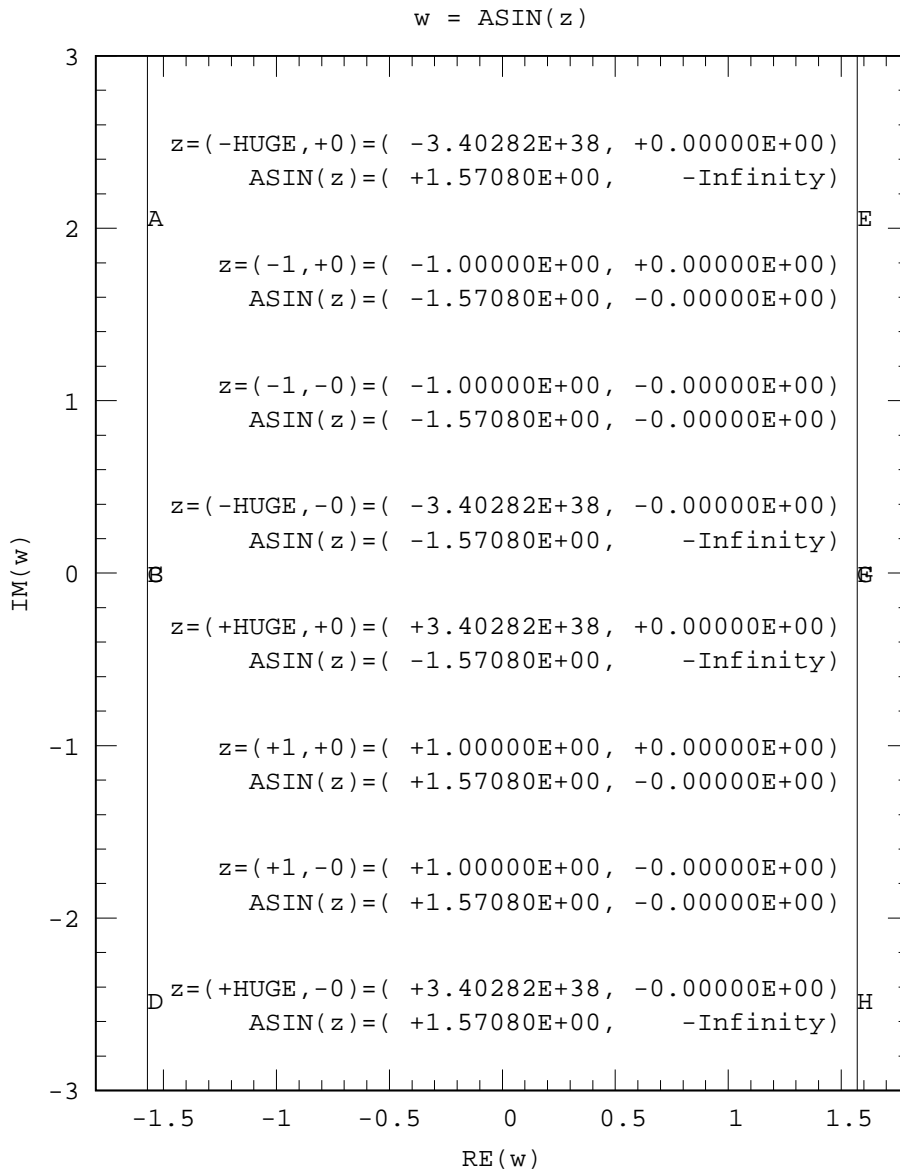
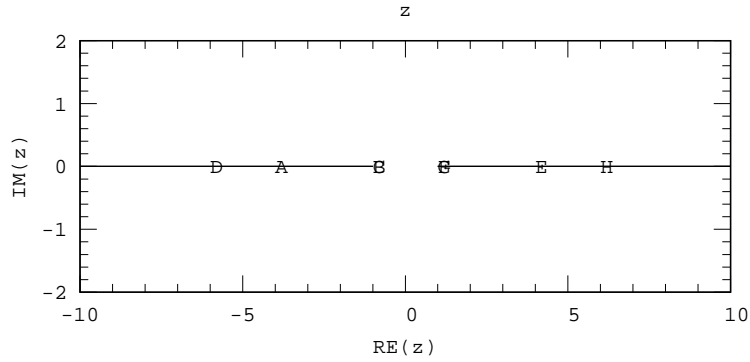
### 2.2.2. SQRT map



### 2.3. ASIN

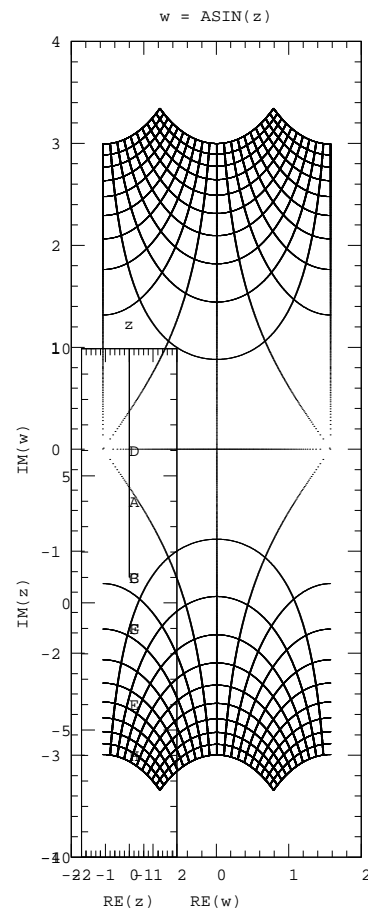
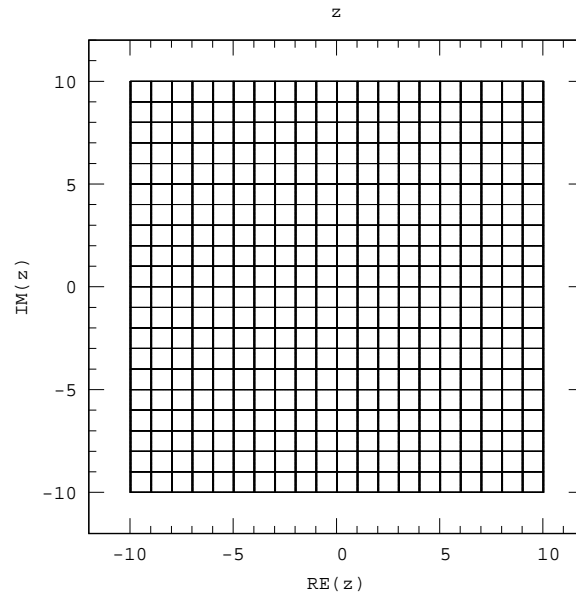
ASIN map is easy to understand. Each cut is opened wider and wider until it becomes a straight line. So the ASIN map is band between  $x = -\pi/2$  and  $x = +\pi/2$ . Along the imaginary axis the band stretches to plus and minus infinity.

2.3.1. ASIN branch cut map





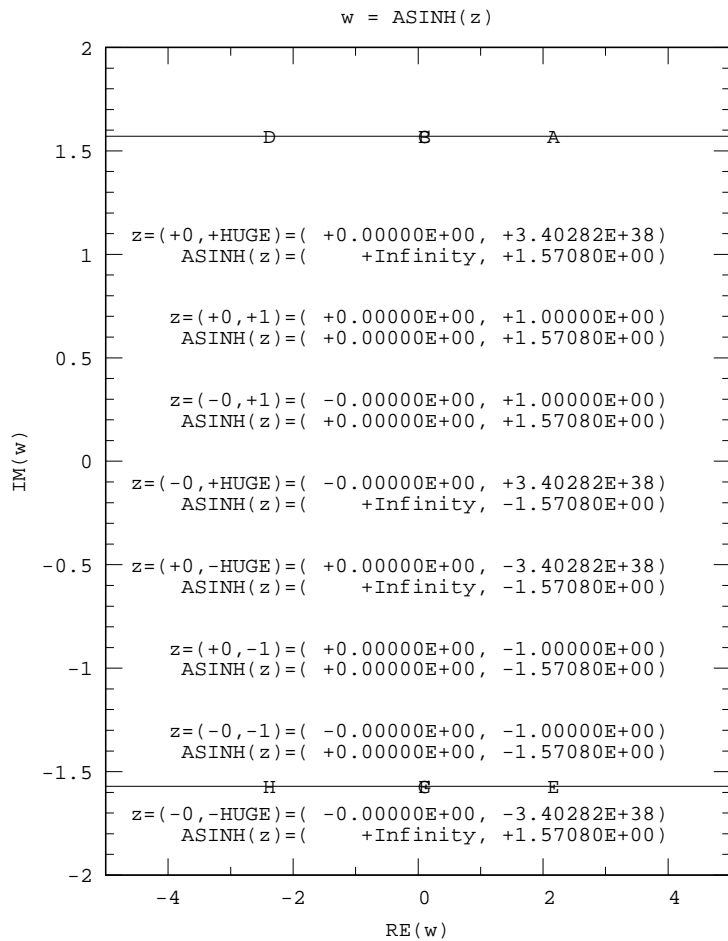
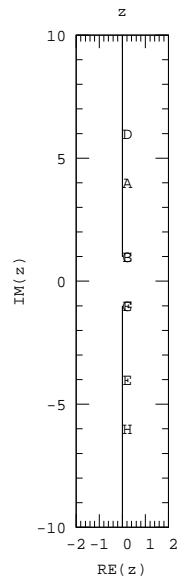
### 2.3.2. ASIN map



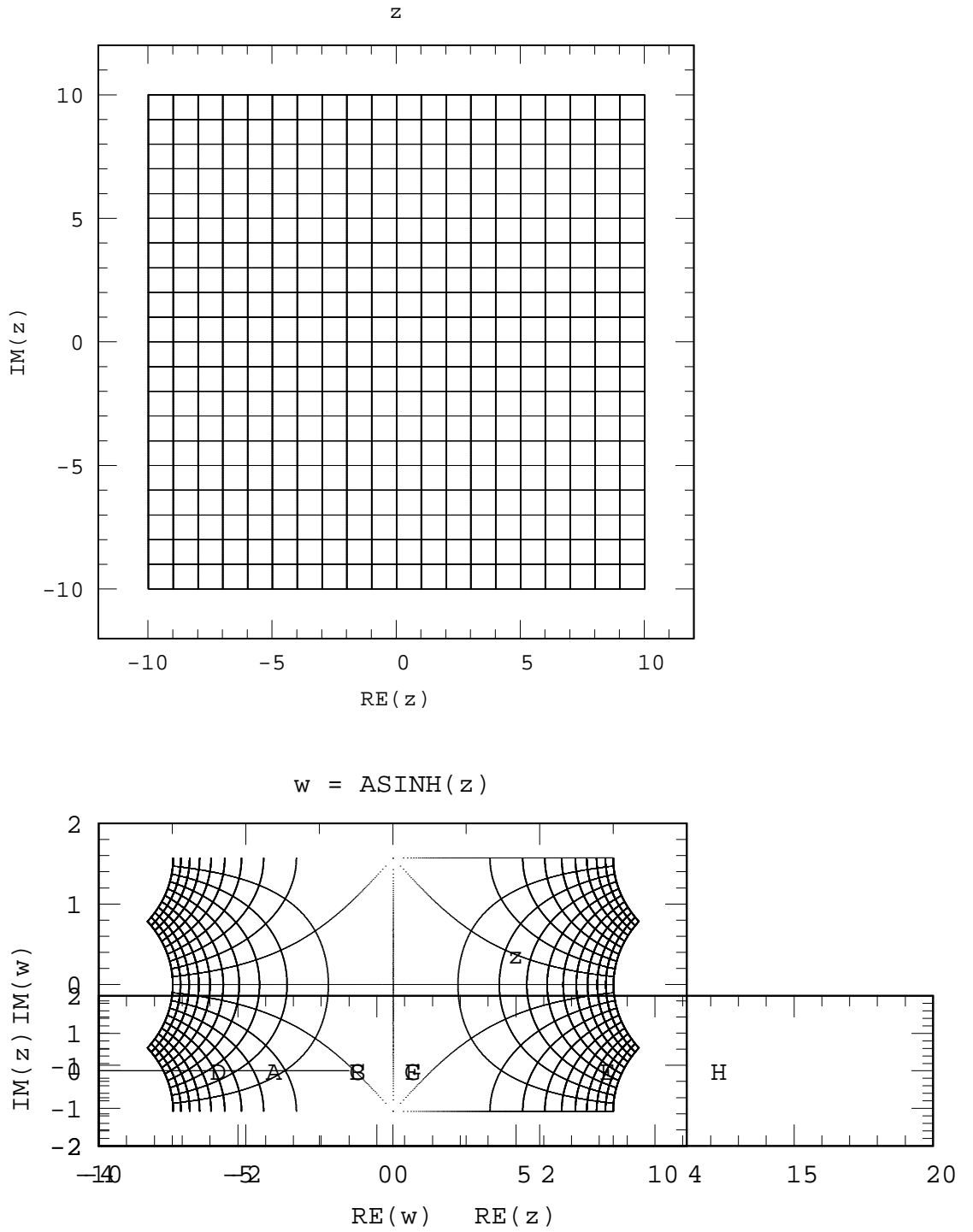
#### 2.4. ASINH

ASINH map is a band between  $y = -\pi/2$  and  $y = \pi/2$ . Along the real axis the band stretches from minus to plus infinity. Points on the branch cut with  $x = +0$  are mapped onto points with  $x \geq 0$ .

2.4.1. ASINH branch cut map



### 2.4.2. ASINH map



## 2.5. ACOS

ACOS is defined everywhere. The branch cuts are on real axis, from  $-\infty$  to  $-1$  and from  $+1$  to  $+\infty$ .

For  $x \leq -1$ , the top boundary of the cut,  $y = +0$ , is mapped onto  $(\pi i, b \leq 0)$ , and the bottom boundary of the cut,  $y = -0$ , is mapped onto  $(\pi i, b \geq 0)$ .

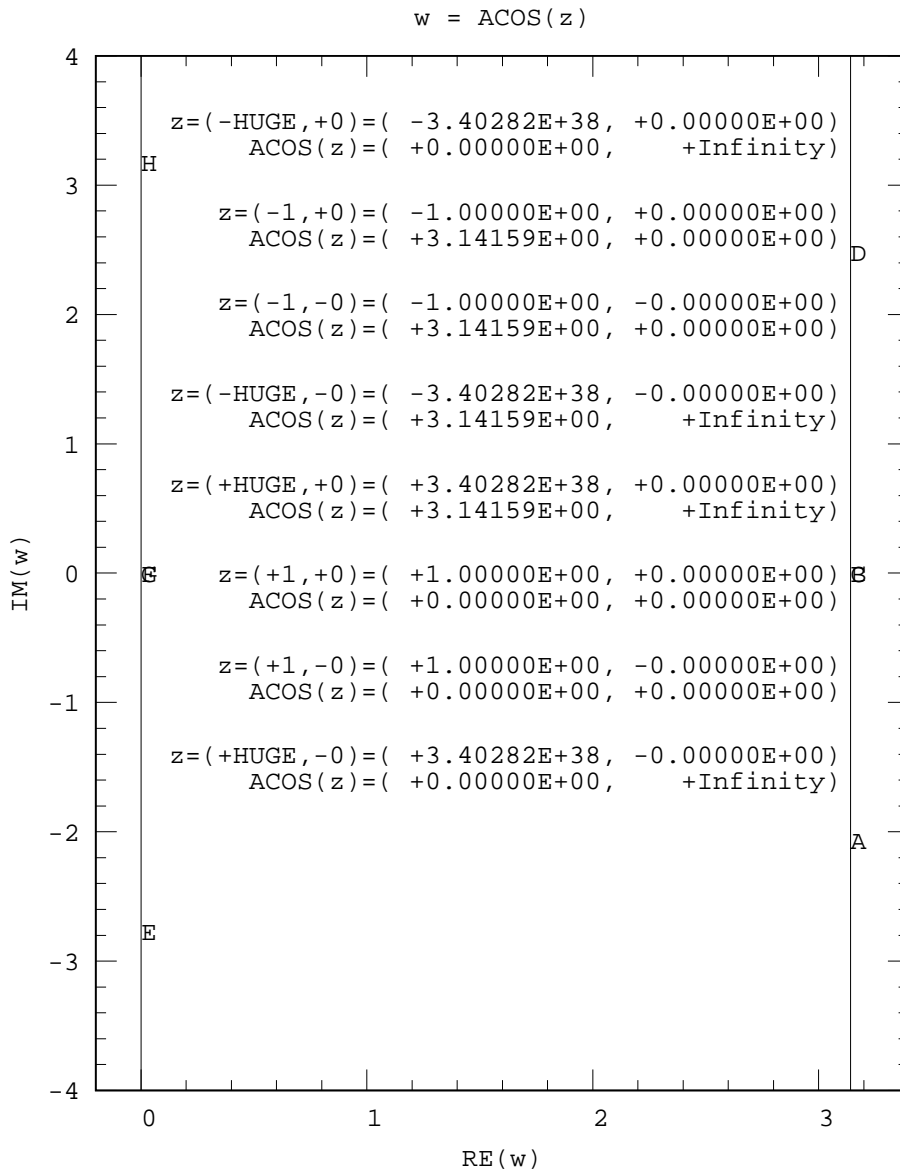
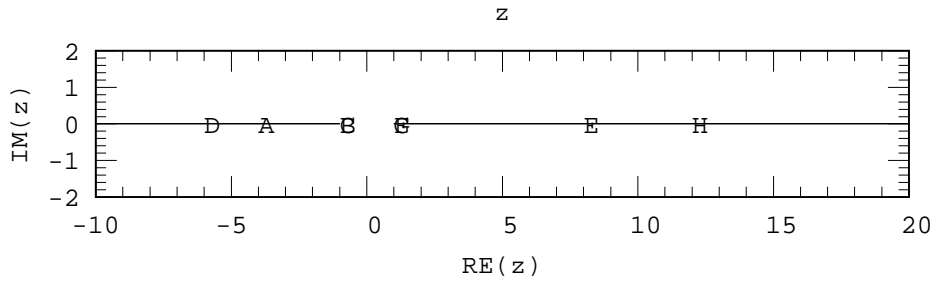
For  $x \geq +1$ , the top boundary of the cut,  $y = +0$ , is mapped onto  $(0, b \leq 0)$ , and the bottom boundary of the cut,  $y = -0$ , is mapped onto  $(0, b \geq 0)$ .

Note that the  $y = 0$  line is mapped onto  $(x, y \leq 0)$ .

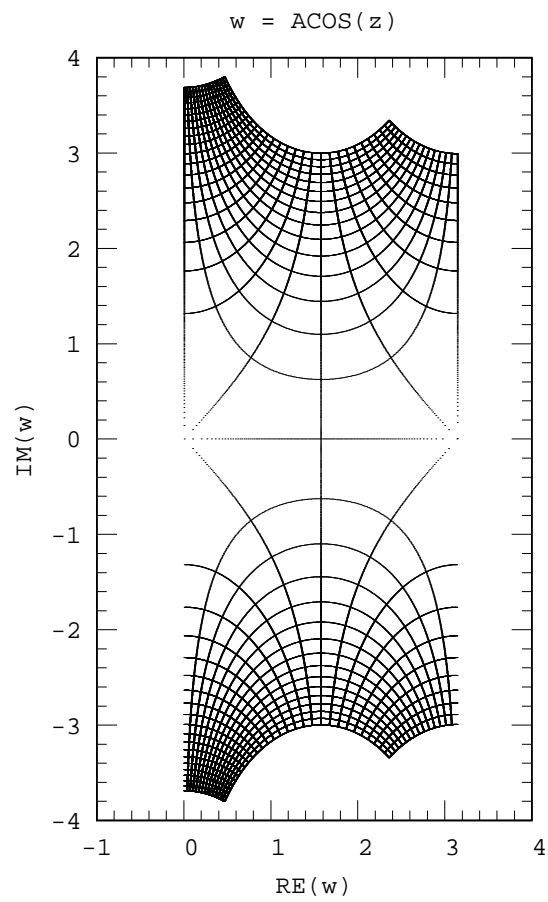
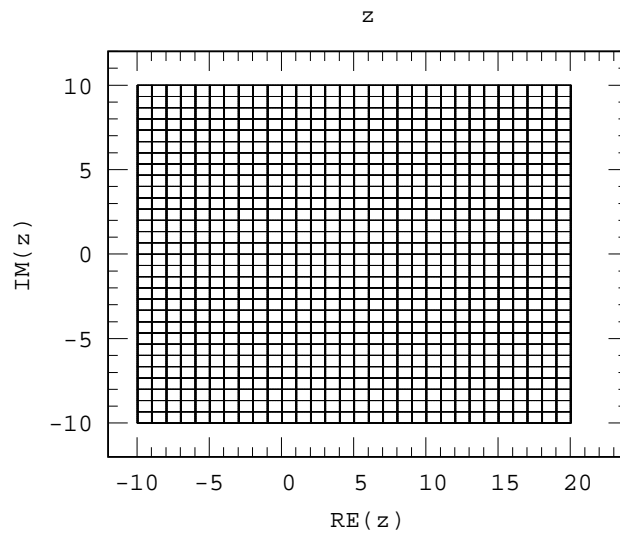
To visualise the ACOS map first visualise the ASIN map. The ACOS map can be obtained from the ASIN map in 3 steps: (1) flip the ASIN map about the real axis, then (2) flip the ASIN map about the imaginary axis, then (3) shift the ASIN map along  $x$  so that the map of the left branch cut starts at  $x = 0$ .

In the second map example, the region of  $z$  which is mapped onto  $w$  is not symmetric with respect to  $x = 0$  axis. This lack of symmetry is used to highlight that points with  $\text{Re}z \geq 0$  are mapped onto points with  $\text{Re}w \leq 0$ .

### 2.5.1. ACOS branch cut map



2.5.2. ACOS map

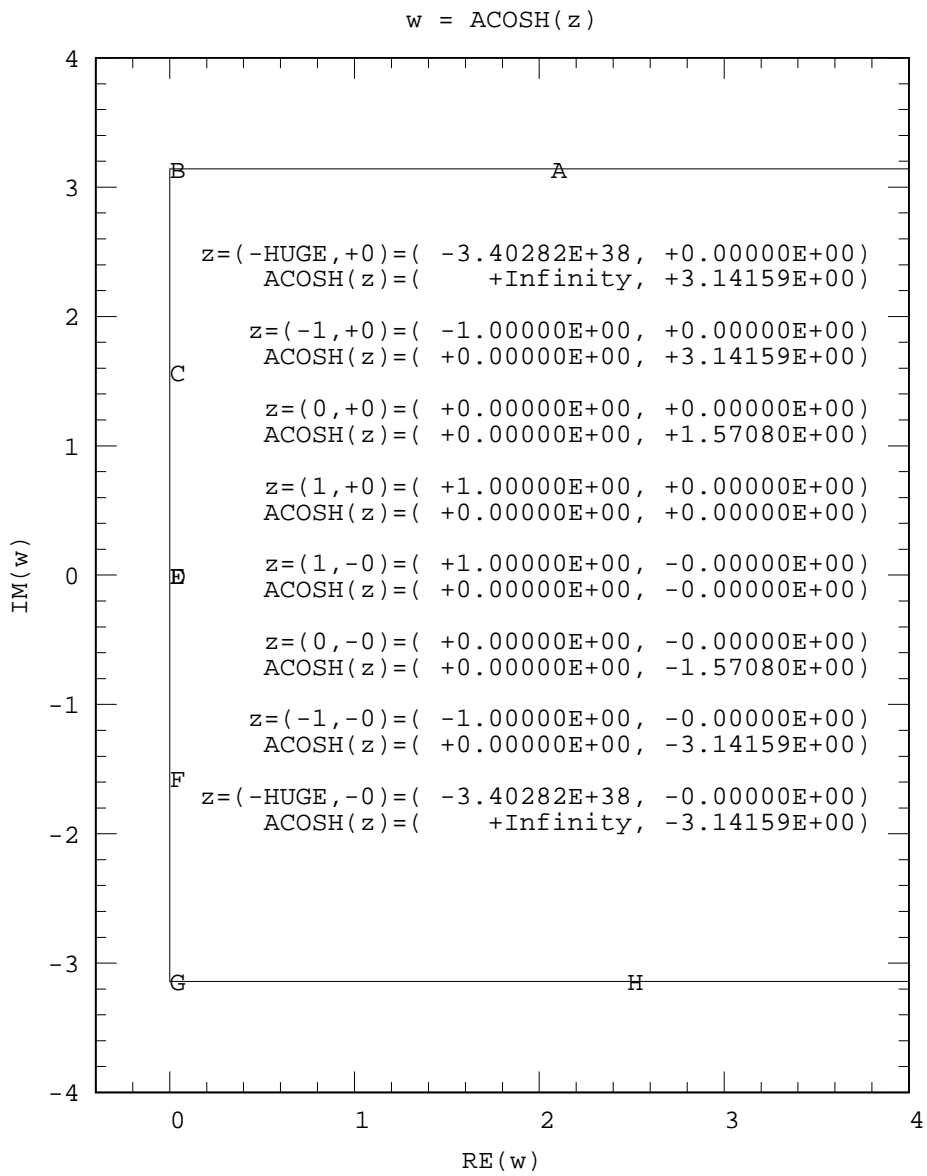
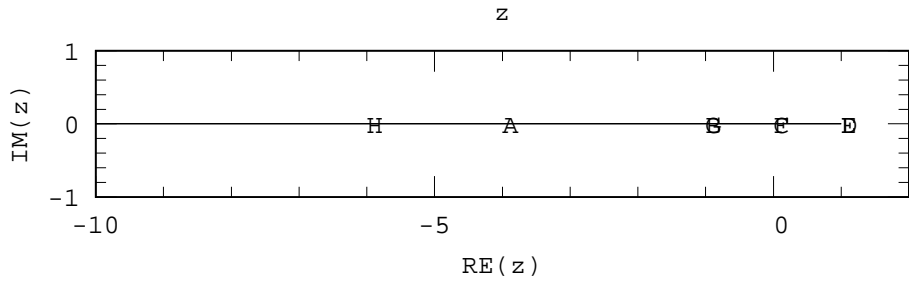


## 2.6. ACOSH

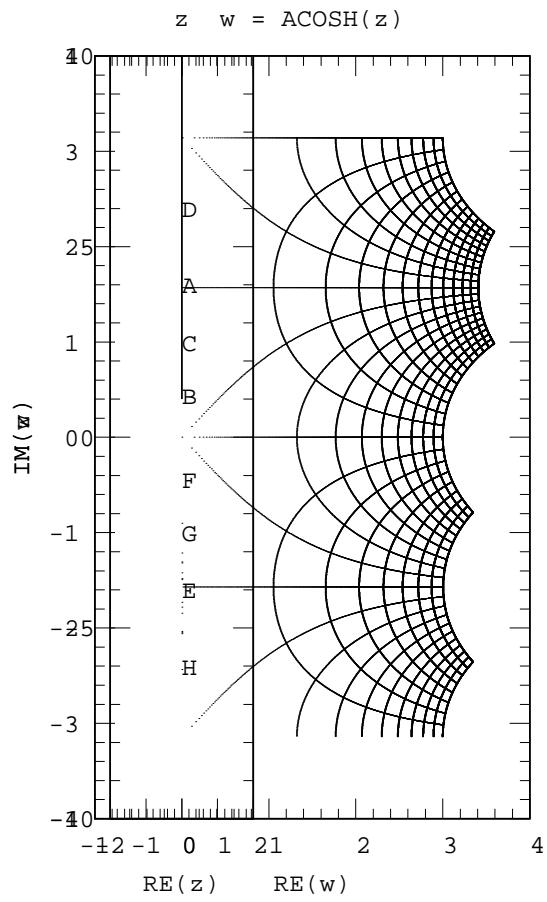
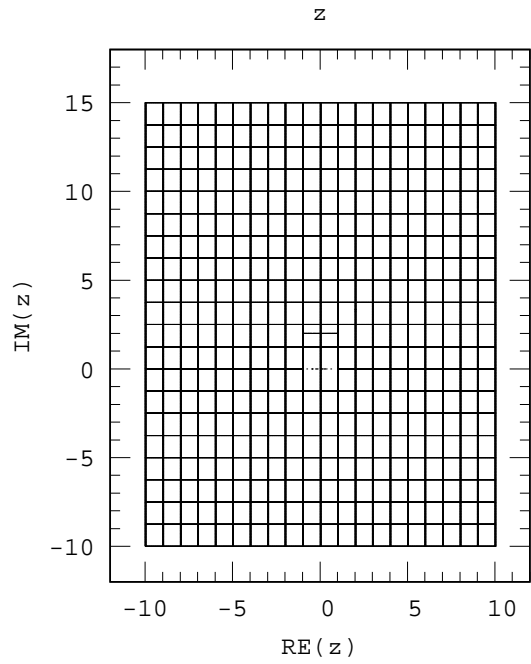
ACOSH has a single branch cut along the real axis from  $x = -1$  to  $-\infty$ . Checking at points  $-HUGE$ ,  $-1$ ,  $0$ ,  $1$  on both sides of the cut. Note that without  $-0$  only the top boundary of the branch cut is mapped. The symmetry is lost.



2.6.1. ACOSH branch cut map



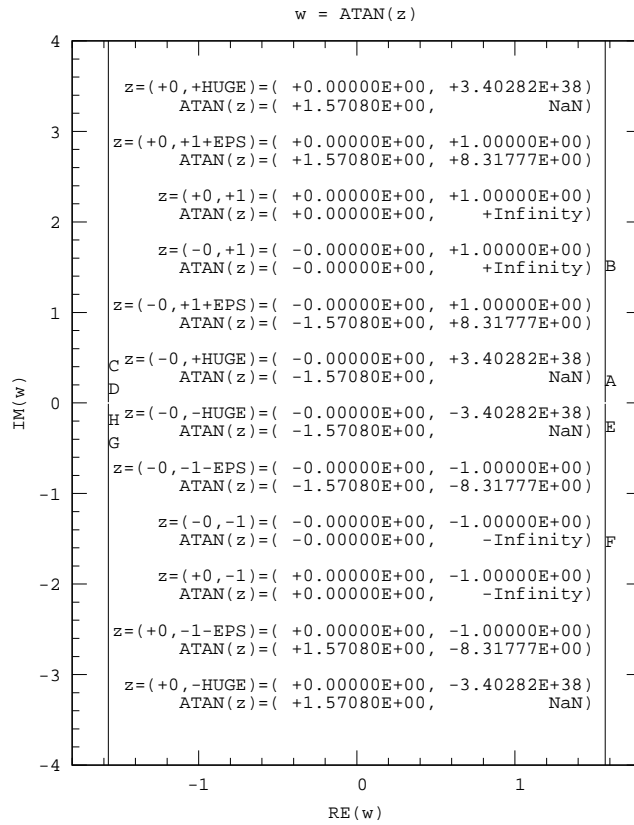
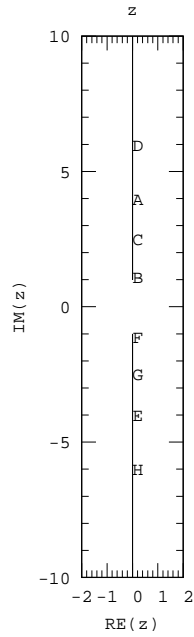
2.6.2. ACOSH map



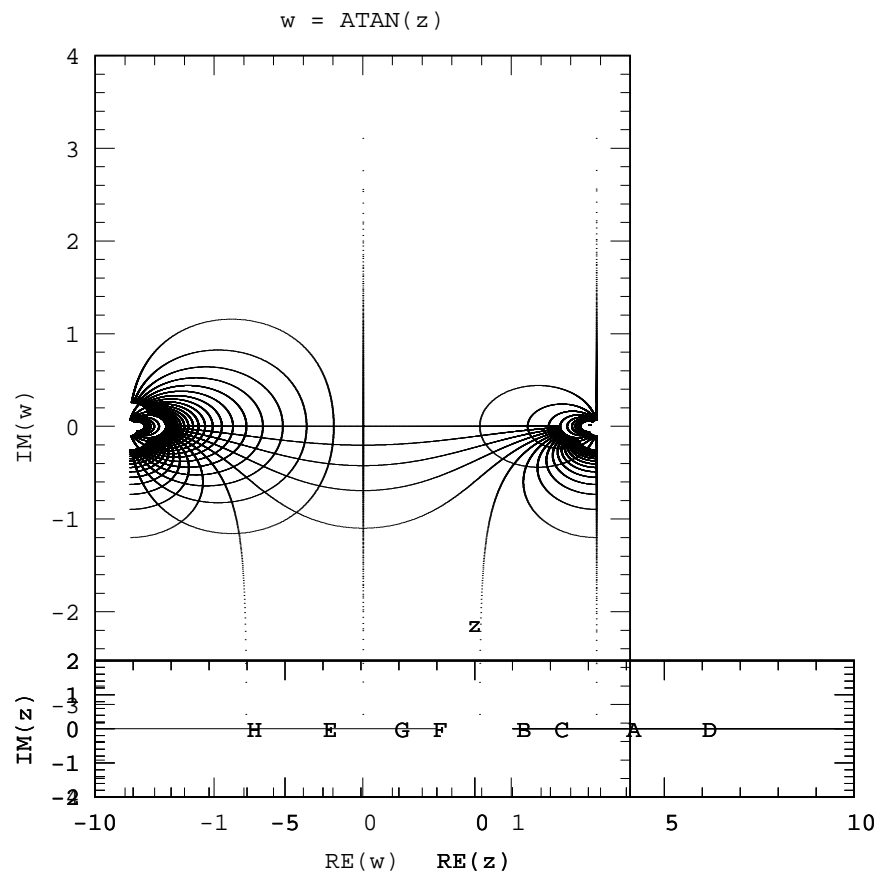
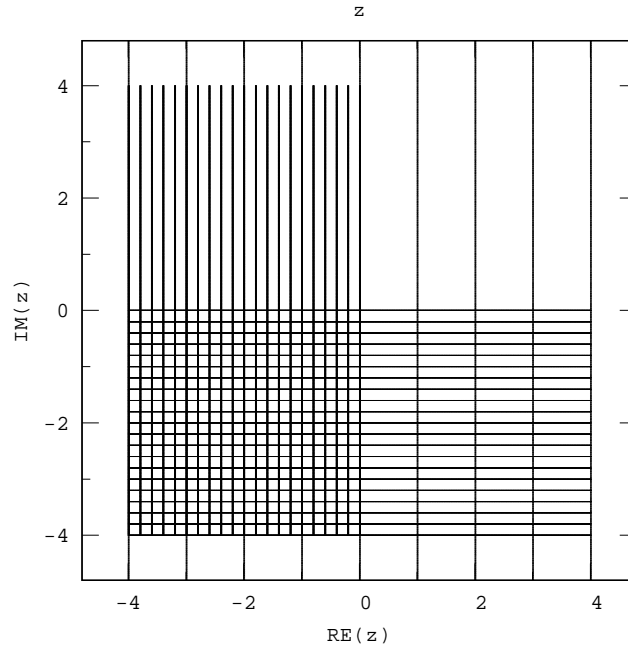
**2.7. ATAN**

ATAN has 2 branch cuts along the imaginary axis, from  $+i$  to  $+\infty$ , and from  $-i$  to  $-\infty$ .

2.7.1. ATAN branch cut map

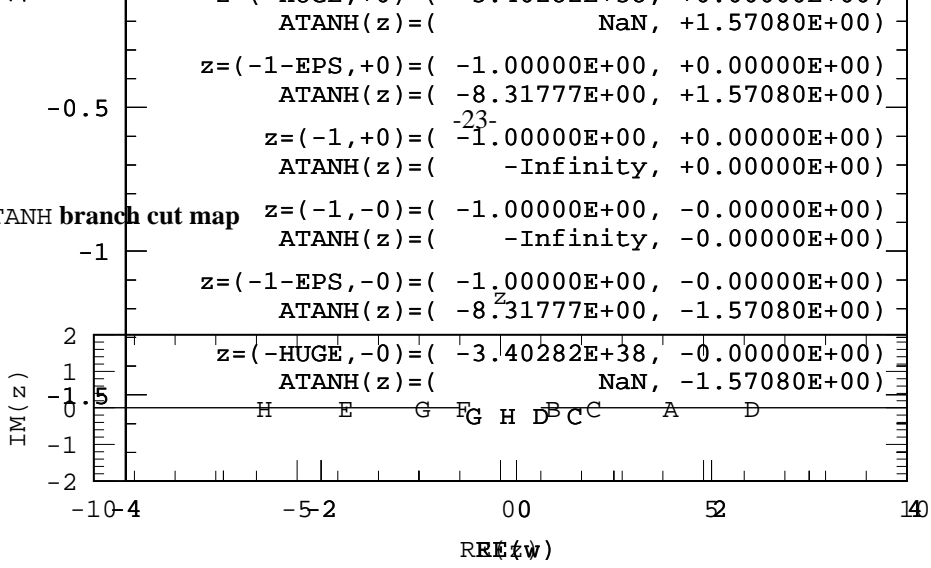


### 2.7.2. ATAN map

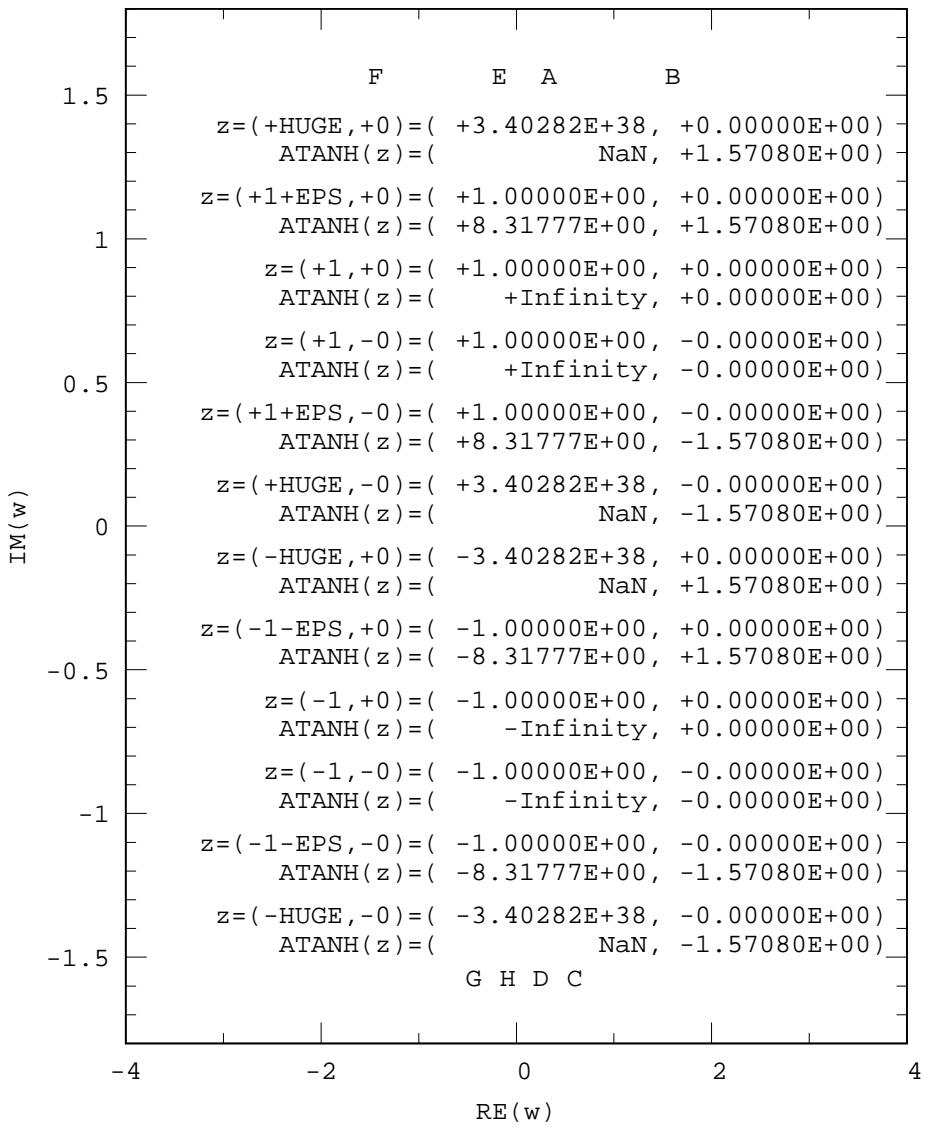


**2.8. ATANH**

2.8.1. ATANH branch cut map



$w = ATANH(z)$



### 2.8.2. ATANH map

